

AVERAGING ALGEBRAS, SCHRÖDER NUMBERS, ROOTED TREES AND OPERADS

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ABSTRACT. In this paper, we study averaging operators from an algebraic and combinatorial point of view. We first construct free averaging algebras in terms of a class of bracketed words called averaging words. We next apply this construction to obtain generating functions in one or two variables for subsets of averaging words when the averaging operator is taken to be idempotent. When the averaging algebra has an idempotent generator, the generating function in one variable is twice the generating function for large Schröder numbers, leading us to give interpretations of large Schröder numbers in terms of bracketed words and rooted trees, as well as a recursive formula for these numbers. We also give a representation of free averaging algebras by unreduced trees and apply it to give a combinatorial description of the operad of averaging algebras.

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1. INTRODUCTION

Let \mathbf{k} be a unitary commutative ring. An averaging operator on a commutative \mathbf{k} -algebra R is a linear operator P satisfying the identity

$$(1) \quad P(fP(g)) = P(f)P(g) \text{ for all } f, g \in R.$$

This operator was already implicitly studied by O. Reynolds [24] in 1895 in turbulence theory under the disguise of a Reynolds operator which is defined by

$$(2) \quad P(fg) = P(f)P(g) + P[(f - P(f))(g - P(g))] \text{ for all } f, g \in R,$$

since an idempotent Reynolds operator is an averaging operator. An important class of such operators used in turbulence theory is the class of averages over one portion of space time of certain vector fields. For example, the time average of a real valued function f defined on time-space

$$f(x, t) \mapsto \bar{f}(x, t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x, t + \tau) d\tau,$$

is such an operator.

In the 1930s, the notion of averaging operator was explicitly defined by Kolmogoroff and Kampé de Fériet [16, 20]. Then G. Birkhoff [5] continued its study and showed that a positive bounded projection in the Banach algebra $C(X)$, the algebra of scalar valued continuous functions on a compact Hausdorff space X , onto a fixed range space is an idempotent averaging operator. In 1954, S. T. C. Moy [29] made the connection between averaging operators and conditional expectation. Furthermore, she studied the relationship between integration theory and averaging operators in turbulence theory and probability. Then her results were extended by G. C. Rota [25]. During the same period, the idempotent averaging operators on $C_\infty(X)$, the algebra of all real valued continuous functions on a locally compact Hausdorff space X that vanish at the infinity, were characterized by J. L. Kelley [17].

Later on, more discoveries of averaging operators on various spaces were made. B. Brainerd [4] considered the conditions under which an averaging operator can be represented as an integration on the abstract analogue of the ring of real valued measurable functions. In 1964, G. C. Rota [26] proved that a continuous Reynolds operator on the algebra $L_\infty(S, \Sigma, m)$ of bounded measurable functions on a measure space (S, Σ, m) is an averaging operator if and only if it has a closed range. J. L. B. Gamlen and J. B. Miller [20, 10] considered averaging operators on *noncommutative* Banach algebras where the averaging identities are defined by Eq. (3). They discussed spectrum and resolvent sets of averaging operators on Banach algebras. N. H. Bong [6] found some connections between the resolvent of a Rota-Baxter operator [3, 12, 27] and that of an averaging operator on complex Banach algebras. In 1986, Huijsmans generalized the work of Kelley to the case of f -algebras. Triki [30, 31] showed that a positive contractive projection on an Archimedean f -algebra is an idempotent averaging operator.

In the last century, most studies on averaging operators had been done for various special algebras, such as function spaces, Banach algebras, and the topics and methods had been largely analytic. In his Ph. D. thesis in 2000 [7], W. Cao studied averaging operators in the general context and from an algebraic point of view. He gave the explicit construction of free commutative averaging algebras and studied the naturally induced Lie algebra structures from averaging operators.

In this century, while averaging operators continued to find many applications in its traditional areas of analysis and applied areas [9], their algebraic study has been deepened and generalized. J. L. Loday [18] defined the diassociative algebra as the enveloping algebra of the Leibniz algebra by analogy with the associative algebra as the enveloping algebra of the Lie algebra. M. Aguiar [1] showed that a diassociative algebra can be derived from an averaging associative algebra by defining two new operations $x \dot{+} y := xP(y)$ and $x \dot{-} y := P(x)y$. An analogue process gives a Leibniz algebra from an averaging Lie algebra by defining a new operation $\{x, y\} := [P(x), y]$ and derives a (left) permutative algebra from an averaging commutative associative algebra. In general, an averaging operator was defined on any binary operad and this kind of process was systematically studied in [22] by relating the averaging actions to a special construction of binary operads called duplicators [11, 23]. Combining the averaging operators actions with the

Rota-Baxter operators [2, 23] actions, we obtained another connection between Rota-Baxter operators and averaging operators: the resulting algebraic structures given by the actions of the two operators are Koszul dual to each other.

These diverse applications and connections of the averaging algebra motivate us to carry out a further algebraic and combinatorial study of the averaging algebra in this paper. It is well known that in the category of any given algebraic structure, the free objects play a central role in studying the other objects. Further the combinatorial nature of the algebraic structure is often revealed by its free objects (see [24] for the Lie algebra case). Thus our first step is to construct free averaging algebras, after presenting some preliminary properties and examples of averaging algebras. This is carried out in Section 2, where the free averaging algebra on a set is realized on the free module on a set of bracketed words composed from the set, called averaging words. In Section 3, we begin our combinatorial investigation by enumerating subsets of averaging words for the free averaging algebra on one generator and when the operator is taken to be idempotent. The generating function from the enumeration of averaging words turns out to be twice the generating function of the large Schröder numbers, revealing the combinatorial nature of averaging algebras. Pursuing this numerical connection of averaging algebra with large Schröder numbers further allows us to find two applications of averaging algebras to large Schröder numbers. We obtained two interpretations of large Schröder numbers, one in terms of averaging words, another in terms of decorated rooted trees. We also obtain a recursive formula for large Schröder numbers from such a formula arising from the study of averaging words. In Section 4, we identify the set of averaging words with a special class of unreduced planar trees and applied it to give a combinatorial description of the operad of averaging algebras.

2. PROPERTIES AND FREE OBJECTS OF AVERAGING ALGEBRAS

Convention. Throughout this paper, all algebras are taken to be nonunitary unless otherwise specified.

In this section, we first give some properties and examples of averaging operators. We then give an explicit construction of the free averaging algebra on a non-empty set X .

2.1. Definitions and properties. An averaging operator in the noncommutative context is given as follows.

Definition 2.1. A linear operator P on a \mathbf{k} -algebra R is called an **averaging operator** if

$$(3) \quad P(x)P(y) = P(xP(y)) = P(P(x)y) \text{ for all } x, y \in R.$$

A \mathbf{k} -algebra R together with an averaging operator P on R is called an **averaging algebra**.

It is well-known and easily checked that an idempotent operator is an averaging operator if and only if it is a Reynolds operator defined in Eq. (2). There is also a close relationship between averaging operators and Rota-Baxter operators (of weight zero) on an algebra R . The latter operator is defined by the operator equation

$$P(x)P(y) = P(P(x)y) + P(xP(y)) \text{ for all } x, y \in R$$

and has played important role in mathematics and physics [3, 12, 27].

Note that an averaging operator is a set operator in the sense that, for any semigroup (G, \cdot) , it makes sense to define an averaging operator on G to be a map $P : G \rightarrow G$ such that

$$P(x) \cdot P(y) = P(x \cdot P(y)) = P(P(x) \cdot y) \text{ for all } x, y \in G.$$

Now let $P : \mathbb{Q}[x] \rightarrow \mathbb{Q}[x]$ be a linear operator such that, for each $n \geq 0$, we have $P(x^n) = \beta(n)x^{\theta(n)}$ with $\beta(n) \in \mathbb{Q}$ and $\theta(n) > 0$. It is shown in [14] that P is a Rota-Baxter operator of weight zero if and only if θ is an averaging operator on the additive semigroup $\mathbb{Z}_{\geq 1}$:

$$\theta(m) + \theta(n) = \theta(m + \theta(n)) = \theta(\theta(m) + n) \text{ for all } m, n \geq 0.$$

In addition to the examples of averaging algebras mentioned in the introduction, we display the following classes of examples. We first give some examples from averaging processes. Note that if P is an averaging operator, then cP is also an averaging operator for any $c \in \mathbf{k}$.

Proposition 2.2. *Let R be a \mathbf{k} -algebra.*

- (a) *Let G be a finite group that acts on R (on the right) and preserves the multiplication of R : $(xy)^g = x^g y^g$ for all $x, y \in R$ and $g \in G$. Then the linear operator*

$$(4) \quad P : R \rightarrow R, \quad x \mapsto \sum_{g \in G} x^g \text{ for all } x \in R,$$

is an averaging operator.

- (b) *Let a be a fixed element in the center of R . Define $P_a(x) := ax$ for all $x \in R$. Then P_a is an averaging operator on R .*

Proof. (a) For any $x, y \in R$, we have

$$P(xP(y)) = \sum_{h \in G} \left(x \sum_{g \in G} y^g \right)^h = \sum_{h \in G} x^h \sum_{g \in G} y^{gh} = \sum_{h \in G} x^h \sum_{g \in G} y^g = P(x)P(y).$$

We similarly have $P(P(x)y) = P(x)P(y)$.

(b) For any $x, y \in R$, we have

$$P(xP(y)) = a(x(ay)) = P(x)P(y), \quad P(P(x)y) = a((ax)y) = (ax)(ay) = P(x)P(y).$$

Thus P is an averaging operator. \square

As an application of Proposition 2.2.(a), consider $F([a, b], \mathbf{k})$, the \mathbf{k} -algebra of \mathbf{k} -valued functions on the interval $[a, b]$, $a < b$. For a fixed positive integer n , define

$$P : F([a, b], \mathbf{k}) \rightarrow F([a, b], \mathbf{k}), \quad f(x) \mapsto \sum_i f\left(x + \frac{i}{b-a}\right),$$

where the sum is over $i \in \mathbb{Z}$ such that $x + \frac{i}{b-a}$ is in $[a, b]$. Then P is an averaging operator on $F([a, b], \mathbb{R})$. Here we take G to be the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acting on $[a, b]$ by permuting the n subintervals $[a + \frac{i}{b-a}, a + \frac{i+1}{b-a})$, $0 \leq i \leq n-1$. This action induces an action of $\mathbb{Z}/n\mathbb{Z}$ on $F([a, b], \mathbf{k})$ and hence Proposition 2.2.(a) applies. When P is replaced by $\frac{1}{n}P$ which makes sense whenever \mathbf{k} contains \mathbb{Q} , then we obtain the usual averaging operator.

As a special case of Proposition 2.2.(b), let G be a finite group and let $\mathbf{k}[G]$ be the group algebra. Then the \mathbf{k} -linear operator

$$P : \mathbf{k}[G] \rightarrow \mathbf{k}[G], \quad g \mapsto \sum_{h \in G} hg = \left(\sum_{h \in G} h \right) g, \text{ for all } g \in G,$$

is an averaging operator since $\sum_{h \in G} h$ is in the center of $\mathbf{k}[G]$.

There are many averaging operators that do not come from an averaging process. A derivation on a \mathbf{k} -algebra R is a linear operator $d : R \rightarrow R$ such that

$$d(xy) = d(x)y + xd(y) \quad \text{for all } x, y \in R.$$

It is immediately checked that a differential operator derivation d with $d^2 = 0$ is an averaging operator.

Birkhoff showed that an averaging operator on a unitary \mathbf{k} -algebra that preserves the identity 1_R must be idempotent: $P^2(x) = P(1_R P(x)) = P(1_R)P(x) = P(x)$ for all $x \in R$ [25]. We next determine the conditions for an idempotent linear operator to be an averaging operator. Recall that there is a bijection

$$\{\text{idempotent linear operators on } R\} \leftrightarrow \{\text{linear decompositions } R = R_0 \oplus R_1\}$$

such that $R_0 = \text{im } P$ and $R_1 = \ker P$. The linear map P corresponding to $R = R_0 \oplus R_1$ is called the **projection onto R_0 along R_1** .

Proposition 2.3. *Let R be a \mathbf{k} -algebra and let $P : R \rightarrow R$ be an idempotent linear map. Let $R = R_0 \oplus R_1$ be the corresponding linear decomposition. Then P is an averaging operator if and only if*

$$(5) \quad R_0 R_0 \subseteq R_0, \quad R_0 R_1 \subseteq R_1, \quad R_1 R_0 \subseteq R_1.$$

Proof. For any $x, y \in R$, denote $x = x_0 + x_1$ and $y = y_0 + y_1$ with $x_i, y_i \in R_i, i = 0, 1$.

Suppose P is an averaging operator. Then from $P(R) = R_0$ and $P(x)P(y) = P(xP(y))$ we obtain $R_0 R_0 \subseteq R_0$. Then we have

$$\begin{aligned} P(x)P(y) &= x_0 y_0, \\ P(P(x)y) &= P(x_0 y_0 + x_0 y_1) = P(x_0 y_0) + P(x_0 y_1) = x_0 y_0 + P(x_0 y_1), \\ P(xP(y)) &= P(x_0 y_0 + x_1 y_0) = P(x_0 y_0) + P(x_1 y_0) = x_0 y_0 + P(x_1 y_0). \end{aligned}$$

Thus from Eq. (3) we obtain $P(x_0 y_1) = P(x_1 y_0) = 0$ for all $x_i, y_i \in R_i, i = 0, 1$. Therefore Eq. (5) holds since $R_1 = \ker P$ by the definition of P .

Conversely, suppose Eq. (5) holds. Then we have

$$P(P(x)y) = P(x_0 y_0 + x_0 y_1) = P(x_0 y_0) + P(x_0 y_1) = x_0 y_0 = P(x)P(y)$$

and similarly $P(xP(y)) = P(x)P(y)$ for all $x, y \in R$. Thus P is an averaging operator. \square

Recall that a \mathbf{k} -superalgebra is a \mathbf{k} -algebra R with a \mathbf{k} -module decomposition $R = R_0 \oplus R_1$ such that $R_i R_j \subseteq R_{i+j}$ where the subscripts are taken modulo 2.

Corollary 2.4. (a) An idempotent algebra endomorphism $P : R \rightarrow R$ is an averaging operator.

In particular, when R is an augmented \mathbf{k} -algebra with the augmentation map $\varepsilon : R \rightarrow \mathbf{k}$, then ε is an averaging operator regarded as a linear operator on R .

(b) Let $R = R_0 \oplus R_1$ be a \mathbf{k} -superalgebra. Then the projection P of R to R_0 along R_1 is an averaging operator on A .

Proof. (a) Let $R_0 := \text{im } P$ and $R_1 := \ker P$. Then we have $R = R_0 \oplus R_1$ and P is the projection to R_0 along R_1 . Since R_1 is an ideal of R , Eq. (5) holds. Hence P is an averaging operator.

(b) This follows since R_0 and R_1 satisfies Eq. (5). \square

2.2. The construction of free averaging algebras. Free commutative averaging algebras were constructed in [7].

Proposition 2.5. ([7, Theorem 2.6]) *Let A be a unitary commutative \mathbf{k} -algebra and let $\mathbf{Sy}(A)$ denote the symmetric algebra on A . On the tensor product algebra $\mathfrak{A} := \mathfrak{A}(A) := A \otimes \mathbf{Sy}(A)$, define the linear operator*

$$P : \mathfrak{A} \longrightarrow \mathfrak{A}, \quad P\left(\sum_i a_i \otimes s_i\right) := \sum_i 1 \otimes (a_i s_i) \quad \text{for all } a_i \in A, s_i \in \mathbf{Sy}(A),$$

where $a_i s_i$ is the product in $\mathbf{Sy}(A)$. Then P is the free commutative averaging algebra on A . When A is taken to be the polynomial algebra $\mathbf{k}[X]$ on a set X , then $\mathfrak{A}(\mathbf{k}[X])$ is the free commutative averaging algebra on X .

We now construct free (noncommutative) averaging algebras. We carry out the construction in this subsection (Theorem 2.12) and provide the proof of the theorem in the next subsection.

2.2.1. A basis of the free averaging algebra. Recall [13] that an operated semigroup (or a semigroup with an operator) is a semigroup U together with an operator $\alpha : U \rightarrow U$. Let X be a given nonempty set. We will first obtain a linear basis of the free averaging algebra on X from the free operated semigroup $\mathfrak{S}(X)$ on X [12, 13].

For any nonempty set Y , let $S(Y)$ be the free semigroup generated by Y and $\lfloor Y \rfloor := \{\lfloor y \rfloor \mid y \in Y\}$ be a replica of Y . Thus $\lfloor Y \rfloor$ is a set that is indexed by Y but disjoint with Y .

Let X be a nonempty set. Define a direct system as follows. Let

$$\mathfrak{S}_0 := S(X), \quad \mathfrak{S}_1 := S(X \sqcup \lfloor \mathfrak{S}_0 \rfloor) = S(X \sqcup \lfloor S(X) \rfloor),$$

with the natural injection

$$i_{0,1} : \mathfrak{S}_0 = S(X) \hookrightarrow \mathfrak{S}_1 = S(X \sqcup \lfloor \mathfrak{S}_0 \rfloor).$$

Inductively assuming that \mathfrak{S}_{n-1} and $i_{n-2,n-1} : \mathfrak{S}_{n-2} \hookrightarrow \mathfrak{S}_{n-1}$ have been obtained for $n \geq 2$, we define

$$\mathfrak{S}_n := S(X \sqcup \lfloor \mathfrak{S}_{n-1} \rfloor)$$

and have the injection

$$\lfloor \mathfrak{S}_{n-2} \rfloor \hookrightarrow \lfloor \mathfrak{S}_{n-1} \rfloor.$$

Then by the freeness of $\mathfrak{S}_{n-1} = S(X \sqcup \lfloor \mathfrak{S}_{n-2} \rfloor)$, we have

$$\mathfrak{S}_{n-1} = S(X \sqcup \lfloor \mathfrak{S}_{n-2} \rfloor) \hookrightarrow S(X \sqcup \lfloor \mathfrak{S}_{n-1} \rfloor) = \mathfrak{S}_n.$$

Finally, define $\mathfrak{S}(X) := \varinjlim \mathfrak{S}_n$ and define an operator on $\mathfrak{S}(X)$ by

$$w \mapsto \lfloor w \rfloor \quad \text{for all } w \in \mathfrak{S}(X).$$

The operator will be used to define an averaging operator later. Elements in $\mathfrak{S}(X)$ are called **bracketed words** on X . Define the depth of $w \in \mathfrak{S}(X)$ to be

$$(6) \quad d(w) := \min\{n \mid w \in \mathfrak{S}_n\}.$$

Taking the limit in $\mathfrak{S}_n = S(X \sqcup \lfloor \mathfrak{S}_{n-1} \rfloor)$, we obtain

$$(7) \quad \mathfrak{S}(X) = S(X \sqcup \lfloor \mathfrak{S}(X) \rfloor).$$

Thus every bracketed word has a unique decomposition, called the **standard decomposition**,

$$(8) \quad w = w_1 w_2 \cdots w_b,$$

where w_i is in X or $[\mathfrak{S}(X)]$ for $i = 1, 2, \dots, b$. Then we define $b = b(w)$ to be the **breadth** of w . Elements of $X \sqcup [\mathfrak{S}(X)]$ are called **indecomposable**. Define the **head index** $h(w)$ of w to be 0 (resp. 1) if w_1 is in X (resp. $[\mathfrak{S}(X)]$). Similarly define the **tail index** $t(w)$ of w to be 0 (resp. 1) if w_b is in X (resp. $[\mathfrak{S}(X)]$). If w is indecomposable, then $h(w) = t(w)$, called the **index** $\text{id}(w)$ of w . Further, by combining strings of indecomposable factors in w that are in X , we obtain the **block decomposition** of w :

$$(9) \quad w = \omega_1 \cdots \omega_r,$$

where each ω_i , $1 \leq i \leq r$, is in either $S(X)$ or $[\mathfrak{S}(X)]$.

For example, for $w = x[y[x]]xy[y]$, we have $d(w) = 2$, $b(w) = 5$, $h(w) = 0$, $t(w) = 1$. Its block decomposition is $w = x[y[x]](xy)[y]$. So $r = 4$.

As is known [8, 12], the free Rota-Baxter algebra on a set is defined on the free \mathbf{k} -module spanned by the set of **Rota-Baxter words** $\mathcal{R}(X) \subseteq \mathfrak{S}(X)$ consisting of bracketed words that do not contain a subword of the form $[u][v]$ where $u, v \in \mathfrak{S}$. It is natural to consider the averaging case in a similar way: Choose the set \mathcal{B} of bracketed words that do not contain a subword of the forms $[u][v]$ and $[[u]v]$, where $u, v \in \mathfrak{S}$. Unfortunately, this restriction is not enough. For example, we have

$$[x[x]^{(2)}] = [x][x]^{(2)} = [[x][x]] = [x[x]]^{(2)}$$

by the axiom of an averaging operator. Here $[]^{(n)}$, $n \geq 0$, denotes the n -th iteration of the operator $[]$. Thus only one of the two elements $[x[x]^{(2)}]$ and $[x[x]]^{(2)}$ can be kept in a basis for the free averaging algebra. This motivates us to give the following definition.

Definition 2.6. Let X be a set. A bracketed word $w \in \mathfrak{S}(X)$ is called an **averaging word** if w does not contain any subword of the form $[u][v]$, $[[u]v]$ or $[u[v]^{(2)}]$ for $u, v \in \mathfrak{S}(X)$. The set of averaging words on X is denoted by $\mathcal{A} = \mathcal{A}(X)$.

For example, $[x]x$, $[x[x]]$, $[x[x]]^{(2)}$, $[x[x]^{(2)}]x[x]$ are averaging words on $\{x\}$.

We will prove in Theorem 2.12 that the free \mathbf{k} -module $\mathbf{k}\mathcal{A}$ spanned by the set $\mathcal{A} = \mathcal{A}(X)$, equipped with a suitably defined multiplication and linear operator is the free averaging algebra on X . In order to carry out the construction and proof, we give the following recursive description of \mathcal{A} .

For any nonempty subsets G, H and H' of $\mathfrak{S}(X)$, denote

$$\begin{aligned} \Lambda(G, H) &:= (\sqcup_{r \geq 1} (G[H])^r) \sqcup (\sqcup_{r \geq 1} ([H]G)^r) \sqcup (\sqcup_{r \geq 0} (G[H])^r G) \sqcup (\sqcup_{r \geq 0} ([H]G)^r [H]), \\ \Lambda^+(G, H, H') &:= (\sqcup_{r \geq 1} (G[H])^{r-1} G[H']) \sqcup (\sqcup_{r \geq 0} (G[H])^r G), \end{aligned}$$

where, for a subset T of $\mathfrak{S}(X)$, $T^r := \{t_1 \cdots t_r \mid t_i \in T, 1 \leq i \leq r\}$ and $T^0 := \mathbf{1}$, the empty word.

We construct direct systems $\{\mathcal{A}_n^+\}_{n \geq 0}$, $\{\tilde{\mathcal{A}}_n^+\}_{n \geq 0}$, $\{\mathcal{A}_n\}_{n \geq 0}$ from $\mathfrak{S}(X)$ by the following recursions. First denote $\mathcal{A}_0 = \mathcal{A}_0^+ = \tilde{\mathcal{A}}_0^+ = S(X)$. Then for $n \geq 0$, define

$$(10) \quad \mathcal{A}_{n+1} = \Lambda(\mathcal{A}_0, \tilde{\mathcal{A}}_n^+),$$

$$(11) \quad \mathcal{A}_{n+1}^+ = \Lambda^+(\mathcal{A}_0, \tilde{\mathcal{A}}_n^+, \mathcal{A}_n^+), \quad \tilde{\mathcal{A}}_{n+1}^+ = \mathcal{A}_{n+1}^+ \sqcup [\tilde{\mathcal{A}}_n^+].$$

We have the following properties on \mathcal{A}_n , \mathcal{A}_n^+ and $\tilde{\mathcal{A}}_n^+$.

Proposition 2.7. For $n \geq 0$, we have

$$(12) \quad \mathcal{A}_n \subseteq \mathcal{A}_{n+1},$$

$$(13) \quad \mathcal{A}_n^+ \subseteq \mathcal{A}_{n+1}^+,$$

$$(14) \quad \tilde{\mathcal{A}}_n^+ \subseteq \tilde{\mathcal{A}}_{n+1}^+.$$

Proof. We prove the inclusions by induction on n . When $n = 0$, by definition, we have

$$\mathcal{A}_0 \subseteq \mathcal{A}_1, \quad \mathcal{A}_0^+ \subseteq \mathcal{A}_1^+, \quad \widetilde{\mathcal{A}}_0^+ \subseteq \widetilde{\mathcal{A}}_1^+.$$

Suppose that the inclusions in Eqs.(12)–(14) hold for $n = k \geq 0$, that is $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$, $\mathcal{A}_k^+ \subseteq \mathcal{A}_{k+1}^+$ and $\widetilde{\mathcal{A}}_k^+ \subseteq \widetilde{\mathcal{A}}_{k+1}^+$. Consider the case $n = k + 1$. Then we immediately have

$$\begin{aligned} \mathcal{A}_{k+1} &= \Lambda(\mathcal{A}_0, \widetilde{\mathcal{A}}_k^+) \subseteq \Lambda(\mathcal{A}_0, \widetilde{\mathcal{A}}_{k+1}^+) = \mathcal{A}_{k+2}, \\ \mathcal{A}_{k+1}^+ &= \Lambda^+(\mathcal{A}_0, \widetilde{\mathcal{A}}_k^+, \mathcal{A}_k^+) \subseteq \Lambda^+(\mathcal{A}_0, \widetilde{\mathcal{A}}_{k+1}^+, \mathcal{A}_{k+1}^+) = \mathcal{A}_{k+2}^+, \\ \widetilde{\mathcal{A}}_{k+1}^+ &= \mathcal{A}_{k+1}^+ \sqcup \lfloor \widetilde{\mathcal{A}}_k^+ \rfloor \subseteq \mathcal{A}_{k+2}^+ \sqcup \lfloor \widetilde{\mathcal{A}}_{k+1}^+ \rfloor = \widetilde{\mathcal{A}}_{k+2}^+. \end{aligned}$$

These complete the induction. \square

Therefore we can take the direct systems

$$\mathcal{A}_\infty := \bigcup_{n \geq 0} \mathcal{A}_n = \varinjlim \mathcal{A}_n, \quad \mathcal{A}^+ := \bigcup_{n \geq 0} \mathcal{A}_n^+ = \varinjlim \mathcal{A}_n^+, \quad \widetilde{\mathcal{A}}^+ := \bigcup_{n \geq 0} \widetilde{\mathcal{A}}_n^+ = \varinjlim \widetilde{\mathcal{A}}_n^+.$$

Proposition 2.8. *For a given set X , we have $\mathcal{A} = \mathcal{A}_\infty$.*

Because of the proposition, we will omit the notation \mathcal{A}_∞ in the rest of the paper.

Proof. For now we let $\mathcal{A}_{(n)}$, $n \geq 0$, denote the subset of \mathcal{A} consisting of its elements of depth less or equal to n : $\mathcal{A}_{(n)} := \mathcal{A} \cap \mathfrak{S}_n$. Then we have $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}_{(n)}$. So we just need to verify

$$(15) \quad \mathcal{A}_{(n)} = \mathcal{A}_n \text{ for all } n \geq 0.$$

We will prove it by induction on $n \geq 0$.

When $n = 0$, there is nothing to prove since $\mathcal{A}_{(0)} = \mathcal{A}_0 = S(X)$. Assume that Eq. (15) has been verified for $n \leq k$ with $k \geq 0$ and consider the subsets $\mathcal{A}_{(k+1)}$ and \mathcal{A}_{k+1} .

Since $\mathcal{A}_k := \Lambda^+(\mathcal{A}_0, \widetilde{\mathcal{A}}_{k-1}^+)$ is contained in $\mathcal{A}_{(k)} \subseteq \mathcal{A}_{(k+1)}$, the subsets \mathcal{A}_k^+ and $\widetilde{\mathcal{A}}_k^+$ are contained in $\mathcal{A}_{(k+1)}$ by the induction hypothesis. Thus elements in these subsets do not contain elements of the forms excluded in the definition of \mathcal{A} . Further elements in $\widetilde{\mathcal{A}}_k^+$ do not contain elements of the forms $\lfloor u \rfloor v$ and $u \lfloor v \rfloor^{(2)}$. Thus elements in $\lfloor \widetilde{\mathcal{A}}_k^+ \rfloor$ do not contain elements of the form $\lfloor \lfloor u \rfloor v \rfloor$ and $\lfloor u \rfloor \lfloor v \rfloor^{(2)}$. Therefore by the definition of \mathcal{A}_{k+1} in Eq. (10), elements of \mathcal{A}_{k+1} do not contain subwords of the forms excluded in the definition of \mathcal{A} . Since \mathcal{A}_{k+1} is also contained in \mathfrak{S}_{k+1} , we have $\mathcal{A}_{k+1} \subseteq \mathcal{A}_{(k+1)}$.

Conversely, since elements of $\mathcal{A}_{(k+1)}$ have depth less or equal to $k + 1$ and do not contain subwords of the form $\lfloor u \rfloor \lfloor v \rfloor$, by the induction hypothesis we have $\mathcal{A}_{(k+1)} \subseteq \Lambda(\mathcal{A}_0, \mathcal{A}_k)$. Since elements in \mathcal{A}_k are in brackets in $\Lambda(\mathcal{A}_0, \mathcal{A}_k)$ and elements of $\mathcal{A}_{(k+1)}$ cannot contain elements of the forms $\lfloor \lfloor u \rfloor v \rfloor$ and $\lfloor u \rfloor \lfloor v \rfloor^{(2)}$, we have

$$\mathcal{A}_{(k+1)} \subseteq \Lambda^+(\mathcal{A}_0, \widetilde{\mathcal{A}}_k^+) = \mathcal{A}_{k+1}.$$

This completes the induction. \square

Remark 2.9. If v is in $\lfloor \widetilde{\mathcal{A}}^+ \rfloor$, then there is unique $s \geq 1$ such that $v = \lfloor v' \rfloor^{(s)}$ with $v' \in \mathcal{A}^+$. Thus if $v' = v'_1 \cdots v'_n$ is in standard form, then $v'_1 = x$ and v'_n is either x or $\lfloor \widetilde{v}'_n \rfloor$ with $\widetilde{v}'_n \in \mathcal{A}^+$ when $n \geq 2$.

Taking the limit on both sides of Eq. (10), we obtain $\mathcal{A} = \Lambda(\mathcal{A}_0, \widetilde{\mathcal{A}}^+)$. Thus in the block decomposition $w = \omega_1 \cdots \omega_r$ of $w \in \mathcal{A}$, the elements $\omega_1, \dots, \omega_r$ are alternatively in $\mathcal{A}_0 = S(X)$ and $\lfloor \widetilde{\mathcal{A}}^+ \rfloor$. We show next that these are also sufficient conditions for w to be in \mathcal{A} .

- Lemma 2.10.** (a) Let $w = \omega_1 \cdots \omega_r$ be the block decomposition of $w \in \mathfrak{S}(X)$ in Eq. (9). Then w is an averaging word if and only if w is in $\Lambda(X, \mathfrak{S}(X))$ and each $\omega_i, 1 \leq i \leq r$, is an averaging word and $\text{id}(w_i) \neq \text{id}(w_{i+1})$ for $1 \leq i \leq r-1$.
- (b) Let $w = w_1 \cdots w_b$ be the standard decomposition of $w \in \mathfrak{S}(X)$ in Eq. (8). If w is an averaging word, then each $w_i, 1 \leq i \leq b$, is an averaging word. Further, if w is an averaging word and $w = uv$ with $u, v \in \mathfrak{S}(X)$, then u and v are averaging words.

Proof. (a) Let $w \in \mathfrak{S}(X)$ with block decomposition $w = \omega_1 \cdots \omega_r$ be an averaging word. Then by the definition of an averaging word, w does not contain any of the subwords excluded in the definition of averaging words. Thus each $\omega_i, 1 \leq i \leq r$, does not contain these subwords and hence is an averaging word.

Conversely, let $w = \omega_1 \cdots \omega_r$ be in $\Lambda(X, \mathfrak{S}(X))$ and $\omega_1, \dots, \omega_r$ be averaging words. Since w is in \mathfrak{S}_k for some $k \geq 0$, we just need to prove that w is in \mathcal{A} by induction on $k \geq 0$. When $k = 0$, we have $\mathfrak{S}_0 = S(X)$ which is in \mathcal{A} . Hence the claim holds. Assume that the claim holds for all $w \in \mathfrak{S}_k$ where $k \geq 0$ and consider $w \in \mathfrak{S}_{k+1}$. Thus w is in $\Lambda(X, \mathfrak{S})$ and each $\omega_i, 1 \leq i \leq r$, is in \mathcal{A}_{k+1} . The second condition means that each ω_i is either in $S(X)$ or in $[\widetilde{\mathcal{A}}_k^+]$. Then the first condition means that w is in $\Lambda(S(X), \widetilde{\mathcal{A}}_k^+) = \mathcal{A}_{k+1}$. This completes the induction.

(b) Both statements follow directly from the definition of averaging words. \square

2.2.2. Construction of the product and operator. Let X be a set and let $\mathbf{k}\mathcal{A}$ to be the free \mathbf{k} -module generated by $\mathcal{A} := \mathcal{A}(X)$. To define a multiplication \diamond on $\mathbf{k}\mathcal{A}$, we first define $u \diamond v$ for two words u and v in \mathcal{A} by induction on the depth $d(u) \geq 0$ of u as follows.

If $d(u) = 0$, then u is in $S(X)$ and the product \diamond is the concatenation. Assume that $u \diamond v$ have been defined for all $u, v \in \mathcal{A}$ with $d(u) \leq k$ where $k \geq 0$. Consider $u, v \in \mathcal{A}$ with $d(u) = k+1$. First consider the case when u and v are indecomposable, namely are in $X \sqcup [\mathcal{A}]$. For $u, v \in [\mathcal{A}]$, rewrite $u = [u]^{(s)}, v = [v']^{(t)}$, where $s, t \geq 1$ while u', v' are in \mathcal{A}^+ as in Remark 2.9 and hence are not in $[\mathcal{A}]$. Then define

$$(16) \quad u \diamond v = \begin{cases} uv, & \text{if } u \text{ or } v \text{ is in } X, \\ [u' \diamond [v']^{(s+t-1)}], & \text{if } u = [u']^{(s)} \text{ and } v = [v']^{(t)}, \end{cases}$$

where $u' \diamond [v']$ is defined by the induction hypothesis since $d(u') = d(u) - s$ is less than $k+1$. Next consider the general case when u and v are in \mathcal{A} . Let $u = u_1 u_2 \cdots u_m, v = v_1 v_2 \cdots v_n$ be their standard decompositions in Eq. (8). Then define

$$(17) \quad u \diamond v = u_1 u_2 \cdots u_{m-1} (u_m \diamond v_1) v_2 \cdots v_n,$$

where $u_m \diamond v_1$ is the concatenation or as defined in Eq. (16).

For example, for $u_1 = x, v_1 = [x], u_2 = [x[x]]^{(2)}$ and $v_2 = [x]^{(3)}$, we have

$$u_1 \diamond v_1 = x[x], \quad u_2 \diamond v_2 = [x[x] \diamond [x]]^{(4)} = [x[x[x]]]^{(4)}.$$

By the concatenation case and Eq. (16), we have

$$(18) \quad h(u) = h(u \diamond v), \quad t(v) = t(u \diamond v).$$

Extending \diamond bilinearly, we obtain a binary operation on $\mathbf{k}\mathcal{A}(X)$. The following properties can be derived from the definition of \diamond directly.

Lemma 2.11. Let $w, w' \in \mathcal{A}$. Then

- (a) $h(w) = h(w \diamond w')$ and $t(w') = t(w \diamond w')$.

(b) If $t(w) \neq h(w')$ or $t(w) = h(w') = 0$, then for any $w'' \in \mathcal{A}$,

$$(19) \quad (ww') \diamond w'' = w(w' \diamond w''),$$

$$(20) \quad w'' \diamond (ww') = (w'' \diamond w)w'.$$

2.2.3. *The construction of the operator.* We next define a linear operator P on $\mathbf{k}\mathcal{A}$. For $u \in \mathcal{A}$, there is some n such that $u \in \mathcal{A}_n$. Recall that

$$\mathcal{A}_n^+ = (\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^{r-1} \mathcal{A}_0[\mathcal{A}_{n-1}^+]) \sqcup (\sqcup_{r \geq 0} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^r \mathcal{A}_0), \quad \widetilde{\mathcal{A}}_n^+ = \mathcal{A}_n^+ \sqcup [\widetilde{\mathcal{A}}_{n-1}^+]$$

and

$$(21) \quad \begin{aligned} \mathcal{A}_n = & (\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^r) \sqcup (\sqcup_{r \geq 1} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r) \\ & \sqcup (\sqcup_{r \geq 0} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^r \mathcal{A}_0) \sqcup (\sqcup_{r \geq 0} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r [\widetilde{\mathcal{A}}_{n-1}^+]). \end{aligned}$$

Since $\widetilde{\mathcal{A}}_{n-1}^+ = \mathcal{A}_{n-1}^+ \sqcup [\widetilde{\mathcal{A}}_{n-2}^+]$, the first of the four disjoint union component becomes

$$\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^r = (\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^{r-1} \mathcal{A}_0[\mathcal{A}_{n-1}^+]) \sqcup (\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^{r-1} \mathcal{A}_0[\widetilde{\mathcal{A}}_{n-2}^+]^{(2)}),$$

of which the first disjoint component is exactly the first disjoint union component of \mathcal{A}_n^+ . Also the third disjoint union component of \mathcal{A}_n is the second disjoint union component of \mathcal{A}_n^+ . By collecting the components of $\widetilde{\mathcal{A}}_n^+$ together in this way, we see that \mathcal{A}_n can be rearranged as

$$(22) \quad \mathcal{A}_n = \widetilde{\mathcal{A}}_n^+ \sqcup (\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^{r-1} \mathcal{A}_0[\widetilde{\mathcal{A}}_{n-2}^+]^{(2)}) \sqcup (\sqcup_{r \geq 1} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r) \sqcup (\sqcup_{r \geq 1} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r [\widetilde{\mathcal{A}}_{n-1}^+]).$$

So any $u \in \mathcal{A}_n$ is in one of the above disjoint components. We accordingly define

$$P_X(u) = \begin{cases} [u], & \text{if } u \in \widetilde{\mathcal{A}}_n^+, \\ [u_1 \diamond [u_2]]^{(s)}, & \text{if } u \in \sqcup_{r \geq 1} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r \text{ with } u = [u_1]^{(s)} u_2, \\ [u'_1 [u'_2]]^{(s)}, & \text{if } u \in \sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+])^r \text{ with } u = u'_1 [u'_2]^{(s)}, s \geq 2, \\ [u_1 \diamond [u_2 [u_3]]]^{(s+t-1)}, & \text{if } u \in \sqcup_{r \geq 1} ([\widetilde{\mathcal{A}}_{n-1}^+] \mathcal{A}_0)^r [\widetilde{\mathcal{A}}_{n-1}^+] \text{ with } u = [u_1]^{(s)} u_2 [u_3]^{(t)}, \end{cases}$$

where u_1, u'_2, u_3 are in \mathcal{A}^+ and $h(u_2) = t(u_2) = 0$. Thus $P_X(u)$ is in \mathcal{A} . Then extending P_X by linearity to a linear operator on $\mathcal{A}(X)$ that we still denote by P_X .

For example, if $u = [x[y]]z$, $v = x[y]^{(2)}$ and $w = [x[y]]z[x]^{(2)}$ then we have

$$P_X(u) = [x[y] \diamond [z]] = [x[y]z], \quad P_X(v) = [x[y]]^{(2)},$$

$$P_X(w) = [x[y] \diamond [z[x]]]^{(2)} = [x[y]z[x]]^{(2)}.$$

Let $j_X : X \longrightarrow S(X) \longrightarrow \mathcal{A} \longrightarrow \mathbf{k}\mathcal{A}$ denote the natural injection from X to $\mathcal{A}(X)$.

Theorem 2.12. *Let X be a non-empty set. Then*

- (a) *The pair $(\mathbf{k}\mathcal{A}, \diamond)$ is an algebra;*
- (b) *the triple $(\mathbf{k}\mathcal{A}, \diamond, P_X)$ is an averaging algebra;*
- (c) *the quadruple $(\mathbf{k}\mathcal{A}, \diamond, P_X, j_X)$ is the free averaging algebra on set X . More precisely, for any averaging algebra B and a map $f : X \longrightarrow B$, there is a unique averaging \mathbf{k} -algebra homomorphism $\tilde{f} : \mathbf{k}\mathcal{A} \longrightarrow B$ such that $f = \tilde{f} \circ j_X$.*

The proof of the theorem is given in the next subsection.

2.3. **The proof of Theorem 2.12.** We now prove Theorem 2.12.

(a) We only need to verify the associativity of \diamond :

$$(23) \quad (W \diamond W') \diamond W'' = W \diamond (W' \diamond W'') \quad \text{for all } W, W', W'' \in \mathcal{A}.$$

For this we proceed by induction on the depth $d(W) \geq 0$ of W . If $d(W) = 0$, then W is in $S(X)$. Then $W \diamond W' = WW'$ and $W \diamond (W' \diamond W'') = W(W' \diamond W'')$ and the associativity follows from the definition of \diamond in Eqs. (16) and (17). Suppose Eq. (23) has been verified for all $W, W', W'' \in \mathcal{A}$ with $d(W) \leq k$ for $k \geq 0$ and consider $W, W', W'' \in \mathcal{A}$ with $d(W) = k + 1$. We consider three cases.

Case I. Suppose $t(W) \neq h(W')$ or $t(W) = h(W') = 0$: Then by Lemma 2.11, we have

$$(24) \quad (W \diamond W') \diamond W'' = (WW') \diamond W'' = W(W' \diamond W'') = W \diamond (W' \diamond W'').$$

Case II. Suppose $t(W') \neq h(W'')$ or $t(W') = h(W'') = 0$: This case is proved similarly.

Case III: Suppose $t(W) = h(W') = 1$ and $t(W') = h(W'') = 1$: We divide this case into the following four subcases.

(i) **Suppose $b(W') \geq 2$:** Then $W' = w'_1 w'_2$ with $w'_1, w'_2 \in \mathcal{A}$ and either $t(w'_1) \neq h(w'_2)$ or $t(w'_1) = h(w'_2) = 0$. Then we have

$$\begin{aligned} (W \diamond W') \diamond W'' &= (W \diamond (w'_1 w'_2)) \diamond W'' \\ &= ((W \diamond w'_1) w'_2) \diamond W'' \quad (\text{by Eq. (20)}) \\ &= (W \diamond w'_1)(w'_2 \diamond W'') \quad (\text{by Eq. (19)}) \\ &= W \diamond (w'_1 \diamond (w'_2 \diamond W'')) \quad (\text{by Case I}) \\ &= W \diamond ((w'_1 w'_2) \diamond W'') \quad (\text{by Eq. (24)}) \\ &= W \diamond (W' \diamond W''). \end{aligned}$$

(ii) **Suppose $b(W) \geq 2$:** Then $W = w_1 w_2$ with $w_1 \in \mathcal{A}$, $b(w_2) = 1$ and either $t(w_1) \neq h(w_2)$ or $t(w_1) = h(w_2) = 0$. By Eq. (24), we have

$$(W \diamond W') \diamond W'' = ((w_1 w_2) \diamond W') \diamond W'' = (w_1(w_2 \diamond W')) \diamond W'' = w_1((w_2 \diamond W') \diamond W'')$$

and

$$W \diamond (W' \diamond W'') = (w_1 w_2) \diamond (W' \diamond W'') = w_1(w_2 \diamond (W' \diamond W'')).$$

Thus $(W \diamond W') \diamond W'' = W \diamond (W' \diamond W'')$ holds if and only if $(w_2 \diamond W') \diamond W'' = w_2 \diamond (W' \diamond W'')$ holds. Therefore this case is reduced to the case when $b(W) = 1$ in Subcase (iv).

(iii) **Suppose $b(W'') \geq 2$:** Then $W'' = w''_1 w''_2$ with $w''_2 \in \mathcal{A}$, $b(w''_1) = 1$ and either $t(w''_1) \neq h(w''_2)$ or $t(w''_1) = h(w''_2) = 0$. Similarly to the case when $b(W) \geq 2$, we obtain $(W \diamond W') \diamond W'' = W \diamond (W' \diamond W'')$ holds if and only if $(W \diamond W') \diamond w''_1 = W \diamond (W' \diamond w''_1)$ holds. Thus this case is also reduced to the case when $b(W'') = 1$ in Subcase (iv).

In summary we have reduced the proof of Case III to the proof of the following special case:

(iv) **Suppose $b(W) = b(W') = b(W'') = 1$:** Then all the three words are in $[\widetilde{\mathcal{A}}^+]$. Then $W = \lfloor w \rfloor^{(r)}$, $W' = \lfloor w' \rfloor^{(s)}$, $W'' = \lfloor w'' \rfloor^{(t)}$, where $r, s, t \geq 1$, $w, w', w'' \in \mathcal{A}^+$ and $d(W) = k + 1$. We have

$$(W \diamond W') \diamond W'' = (\lfloor w \diamond \lfloor w' \rfloor \rfloor^{(r+s-1)}) \diamond \lfloor w'' \rfloor^{(t)} = \lfloor (w \diamond \lfloor w' \rfloor) \diamond \lfloor w'' \rfloor \rfloor^{(r+s+t-2)},$$

$$W \diamond (W' \diamond W'') = \lfloor w \rfloor^{(r)} \diamond (\lfloor w' \diamond \lfloor w'' \rfloor \rfloor^{(s+t-1)}) = \lfloor w \diamond (\lfloor w' \diamond \lfloor w'' \rfloor \rfloor) \rfloor^{(r+s+t-2)}.$$

Since $d(w) = k + 1 - s$, by the induction hypothesis we have

$$(w \diamond [w']) \diamond [w''] = w \diamond ([w'] \diamond [w'']) = w \diamond ([w' \diamond [w'']])$$

and then $(W \diamond W') \diamond W'' = W \diamond (W' \diamond W'')$.

This completes the inductive proof of Eq. (23).

(b) We only need to verify the equations

$$(25) \quad P_X(u) \diamond P_X(v) = P_X(P_X(u) \diamond v), \quad P_X(u) \diamond P_X(v) = P_X(u \diamond P_X(v)) \quad \text{for all } u, v \in \mathcal{A}.$$

We first recall by Remark 2.9 that for any $u, v \in \mathcal{A}$, there exist unique $u', v' \in \mathcal{A}^+$ and $s, t \geq 1$ such that

$$(26) \quad P_X(u) = [u']^{(s)}, P_X(v) = [v']^{(t)}$$

Then we have

$$(27) \quad P_X(u) \diamond P_X(v) = [u']^{(s)} \diamond [v']^{(t)} = [u' \diamond v']^{(s+t-1)}.$$

We verify the first equation in Eq. (25) by considering the two cases when $h(v) = 0, 1$.

Case 1. Suppose $h(v) = 0$: When $t(v) = 0$, we have $v \in \mathcal{A}^+$. Then $P_X(v) = [v] = [v']^{(t)}$ in Eq. (26) with $v' = v$ and $t = 1$. When $t(v) = 1$, we rewrite v as $v_1[v_2]^{(\ell)}$, where $v_2 \in \mathcal{A}^+$. Then $P_X(v) = [v_1[v_2]]^{(\ell)} = [v']^{(t)}$ with $v' = v_1[v_2]$ and $t = \ell$. Then by Eq. (27), we have

$$P_X(P_X(u) \diamond v) = P_X([u']^{(s)} v) = \begin{cases} [u' \diamond [v]]^{(s)}, & t(v) = 0 \\ [u' \diamond [v_1[v_2]]]^{(s+\ell-1)}, & t(v) = 1 \end{cases} = [u' \diamond [v']]^{(s+t-1)} = P_X(u) \diamond P_X(v).$$

Case 2. Suppose $h(v) = 1$: When $t(v) = 0$, we rewrite v as $[v_1]^{(\ell)} v_2$, where $v_1 \in \mathcal{A}^+$. Then $P_X(v) = [v_1 \diamond [v_2]]^{(\ell)} = [v']^{(t)}$ with $v' = v_1 \diamond [v_2]$ and $t = \ell$. When $t(v) = 1$, we rewrite v as $[v_1]^{(\ell)} v_2 [v_3]^{(q)}$. Then $P_X(v) = [v_1 \diamond [v_2[v_3]]]^{(\ell+q-1)} = [v']^{(t)}$ with $v' = v_1 \diamond [v_2[v_3]]$ and $t = \ell + q - 1$. Then by Eq. (27), we have

$$\begin{aligned} P_X(P_X(u) \diamond v) &= P_X([u']^{(s)} \diamond v) \\ &= \begin{cases} P_X([u' \diamond [v_1]]^{(s+\ell-1)} v_2), & t(v) = 0 \\ P_X([u' \diamond [v_1]]^{(s+\ell-1)} v_2 [v_3]^{(q)}), & t(v) = 1 \end{cases} \\ &= \begin{cases} [u' \diamond [v_1]] \diamond [v_2]^{(s+\ell-1)}, & t(v) = 0 \\ [u' \diamond [v_1]] \diamond [v_2[v_3]]^{(s+\ell+q-2)}, & t(v) = 1 \end{cases} \\ &= \begin{cases} [u' \diamond [v_1 \diamond [v_2]]]^{(s+\ell-1)}, & t(v) = 0 \\ [u' \diamond [v_1 \diamond [v_2[v_3]]]^{(s+\ell+q-2)}, & t(v) = 1 \end{cases} \\ &= [u' \diamond [v']]^{(s+t-1)} \\ &= P_X(u) \diamond P_X(v). \end{aligned}$$

Thus the first equation in Eq. (25) is verified.

To verify the second equation in Eq. (25), we also consider the two cases when $h(u)$ is 0 and 1.

Case 1. Suppose $h(u) = 0$: When $t(u) = 0$, we have $u \in \mathcal{A}^+$. So $P_X(u) = [u] = [u']^{(s)}$ as in Eq. (26) with $u' = u$ and $s = 1$. When $t(u) = 1$, we rewrite u as $u_1[u_2]^{(\ell)}$, where $u_2 \in \mathcal{A}^+$. Then $P_X(u) = [u_1[u_2]]^{(\ell)} = [u']^{(s)}$ with $u' = u_1[u_2]$, $s = \ell$. Then by Eq. (27) we have

$$P_X(u \diamond P_X(v)) = P_X(u [v']^{(t)})$$

$$\begin{aligned}
&= \begin{cases} \lfloor u \diamond \lfloor v' \rfloor \rfloor^{(t)}, & t(u) = 0 \\ P_X(u_1 \lfloor u_2 \diamond \lfloor v' \rfloor \rfloor^{(\ell+t-1)}), & t(u) = 1 \end{cases} \\
&= \begin{cases} \lfloor u \diamond \lfloor v' \rfloor \rfloor^{(t)}, & t(u) = 0 \\ \lfloor u_1 \lfloor u_2 \diamond \lfloor v' \rfloor \rfloor \rfloor^{(\ell+t-1)}, & t(u) = 1 \end{cases} \\
&= \begin{cases} \lfloor u \diamond \lfloor v' \rfloor \rfloor^{(t)}, & t(u) = 0 \\ \lfloor u_1 \lfloor u_2 \rfloor \diamond \lfloor v' \rfloor \rfloor^{(\ell+t-1)}, & t(u) = 1 \end{cases} \\
&= \lfloor u' \diamond \lfloor v' \rfloor \rfloor^{(s+t-1)} \\
&= P_X(u) \diamond P_X(v).
\end{aligned}$$

Case 2. Suppose $h(u) = 1$: When $t(u) = 0$, we rewrite u as $\lfloor u_1 \rfloor^{(\ell)} u_2$, $u_1 \in \mathcal{A}^+$. Then $P_X(u) = \lfloor u_1 \diamond \lfloor u_2 \rfloor \rfloor^{(\ell)} = \lfloor u' \rfloor^{(s)}$ as in Eq. (26) with $u' = u_1 \diamond \lfloor u_2 \rfloor$ and $s = \ell$. When $t(u) = 1$, we rewrite u as $\lfloor u_1 \rfloor^{(\ell)} u_2 \lfloor u_3 \rfloor^{(q)}$. Then $P_X(u) = \lfloor u_1 \diamond \lfloor u_2 \lfloor u_3 \rfloor \rfloor \rfloor^{(\ell+q-1)} = \lfloor u' \rfloor^{(s)}$ with $u' = u_1 \diamond \lfloor u_2 \lfloor u_3 \rfloor \rfloor$ and $s = \ell + q - 1$. Then we have

$$\begin{aligned}
P_X(u \diamond P_X(v)) &= P_X(u \diamond \lfloor v' \rfloor^{(t)}) \\
&= \begin{cases} P_X(\lfloor u_1 \rfloor^{(\ell)} u_2 \lfloor v' \rfloor^{(t)}), & t(u) = 0 \\ P_X(\lfloor u_1 \rfloor^{(\ell)} u_2 \lfloor u_3 \diamond \lfloor v' \rfloor \rfloor^{(q+t-1)}), & t(u) = 1 \end{cases} \\
&= \begin{cases} \lfloor u_1 \diamond \lfloor u_2 \lfloor v' \rfloor \rfloor \rfloor^{(\ell+t-1)}, & t(u) = 0 \\ \lfloor u_1 \diamond \lfloor u_2 \lfloor u_3 \diamond \lfloor v' \rfloor \rfloor \rfloor \rfloor^{(\ell+q+t-2)}, & t(u) = 1 \end{cases} \\
&= \begin{cases} \lfloor (u_1 \diamond \lfloor u_2 \rfloor) \diamond \lfloor v' \rfloor \rfloor^{(\ell+t-1)}, & t(u) = 0 \\ \lfloor (u_1 \diamond \lfloor u_2 \lfloor u_3 \rfloor \rfloor) \diamond \lfloor v' \rfloor \rfloor^{(\ell+q+t-1)}, & t(u) = 1 \end{cases} \\
&= \lfloor u' \diamond \lfloor v' \rfloor \rfloor^{(s+t-1)} \\
&= P_X(u) \diamond P_X(v).
\end{aligned}$$

This completes the proof of the second equation in Eq. (25). Therefore P_X is an averaging operator.

(c) Let (B, Q) be an averaging algebra and $*$ be the product in B . Let $f : X \rightarrow B$ be a map. We will construct a \mathbf{k} -linear map $\tilde{f} : \mathbf{k}\mathcal{A} \rightarrow B$ by defining $\tilde{f}(w)$ for $w \in \mathcal{A}$. We achieve this by applying the induction on n for $w \in \mathcal{A}_n$. For $w = x_1 x_2 \cdots x_m \in \mathcal{A}_0 = S(X)$, where $x_i \in X$, $1 \leq i \leq m$, define $\tilde{f}(w) = f(x_1) * f(x_2) * \cdots * f(x_m)$. Suppose $\tilde{f}(w)$ has been defined for $w \in \mathcal{A}_n$ and consider $w \in \mathcal{A}_{n+1}$ which is defined by

$$\mathcal{A}_{n+1} = (\sqcup_{r \geq 1} (\mathcal{A}_0 \lfloor \tilde{\mathcal{A}}_n^+ \rfloor)^r) \sqcup (\sqcup_{r \geq 1} (\lfloor \tilde{\mathcal{A}}_n^+ \rfloor \mathcal{A}_0)^r) \sqcup (\sqcup_{r \geq 0} (\mathcal{A}_0 \lfloor \tilde{\mathcal{A}}_n^+ \rfloor)^r \mathcal{A}_0) \sqcup (\sqcup_{r \geq 0} (\lfloor \tilde{\mathcal{A}}_n^+ \rfloor \mathcal{A}_0)^r \lfloor \tilde{\mathcal{A}}_n^+ \rfloor).$$

If w is in $\sqcup_{r \geq 1} (\mathcal{A}_0 \lfloor \tilde{\mathcal{A}}_n^+ \rfloor)^r$, then $w = \prod_{i=1}^r (w_{2i-1} \lfloor w_{2i} \rfloor)$, where $w_{2i-1} \in \mathcal{A}_0$ and $w_{2i} \in \tilde{\mathcal{A}}_n^+$. By the construction of the multiplication \diamond and the averaging operator P_X , we also can express it by

$$w = \diamond_{i=1}^r (w_{2i-1} \diamond P_X(w_{2i})).$$

Thus there is only one possible way to define $\tilde{f}(w)$ in order for \tilde{f} to be an averaging homomorphism:

$$(28) \quad \tilde{f}(w) = *_{i=1}^r (\tilde{f}(w_{2i-1}) * Q(\tilde{f}(w_{2i}))).$$

$\tilde{f}(w)$ can be similarly defined if w is in the other unions. This proves the existence of f as a map and its uniqueness.

We next prove that the map \bar{f} defined in Eq.(28) is indeed an averaging algebra homomorphism. We will first show that \bar{f} is an algebra homomorphism, that is, for any $W, W' \in \mathcal{A}$,

$$(29) \quad \bar{f}(W \diamond W') = \bar{f}(W) * \bar{f}(W').$$

We will prove Eq. (29) by induction on $\mathbf{b} := b(W) + b(W') \geq 2$.

If $b(W) + b(W') = 2$, then $b(W) = b(W') = 1$. We then prove Eq. (29) by induction on $d(W)$. When $d(W) = 0$, that is $W \in S(X)$. Then $W \diamond W'$ is the concatenation and, by definition,

$$\bar{f}(W \diamond W') = \bar{f}(W) * \bar{f}(W').$$

Suppose Eq. (29) holds for W, W' with $0 \leq d(W) \leq k$ and consider $W, W' \in \mathcal{A}$ with $d(W) = k + 1$. If $d(W') = 0$, a similar argument as in the case of $d(W) = 0$ proves Eq. (29). If $d(W') > 0$, then $W = \lfloor w \rfloor^{(s)}$ and $W' = \lfloor w' \rfloor^{(t)}$ with $w, w' \in \mathcal{A}^+$. Hence

$$\begin{aligned} \bar{f}(W \diamond W') &= \bar{f}(\lfloor w \diamond \lfloor w' \rfloor \rfloor^{(s+t-1)}) \\ &= Q^{s+t-1}(\bar{f}(w \diamond \lfloor w' \rfloor)) \quad (\text{by definition}) \\ &= Q^{s+t-1}(\bar{f}(w) * \bar{f}(\lfloor w' \rfloor)) \quad (\text{by induction hypothesis}) \\ &= Q^{s+t-1}(\bar{f}(w) * Q(\bar{f}(w'))) \quad (\text{by definition}) \\ &= Q^s(\bar{f}(w)) * Q^t(\bar{f}(w')) \quad (Q \text{ is an averaging operator}) \\ &= \bar{f}(\lfloor w \rfloor^{(s)}) * \bar{f}(\lfloor w' \rfloor^{(t)}) \quad (\text{by definition}) \\ &= \bar{f}(W) * \bar{f}(W'). \end{aligned}$$

Now assume that Eq. (29) holds for all $W, W' \in \mathcal{A}$ with $2 \leq \mathbf{b} \leq k$. Consider $W, W' \in \mathcal{A}$ with $\mathbf{b} = k + 1$. If $t(W) \neq h(W')$ or $t(W) = h(W') = 0$, then $W \diamond W'$ is the concatenation. So we have

$$\bar{f}(W \diamond W') = \bar{f}(WW') = \bar{f}(W) * \bar{f}(W')$$

by Eq. (28). Now let $t(W) = h(W') = 1$. Since $k \geq 2$, we have $k + 1 \geq 3$, so at least one of W and W' have breadth ≥ 2 . Without loss of generality, we can assume $b(W) \geq 2$. Then W can be written as $w_1 w_2$ with $b(w_2) = 1$. Thus we have $t(w_2) = t(W) = h(W') = 1$, $t(w_1) = 0$, $h(w_2) = 1$, and $b(w_2) + b(W') < b(W) + b(W') = k + 1$. Applying the associativity of the product and the induction hypothesis, we have

$$\begin{aligned} \bar{f}(W \diamond W') &= \bar{f}((w_1 w_2) \diamond W') \\ &= \bar{f}(w_1(w_2 \diamond W')) \\ &= \bar{f}(w_1 \diamond (w_2 \diamond W')) \\ &= \bar{f}(w_1) * \bar{f}(w_2 \diamond W') \quad (t(w_1) \neq h(w_2)) \\ &= \bar{f}(w_1) * (\bar{f}(w_2) * \bar{f}(W')) \quad (\text{by induction hypothesis}) \\ &= (\bar{f}(w_1) * \bar{f}(w_2)) * \bar{f}(W') \\ &= \bar{f}(w_1 \diamond w_2) * \bar{f}(W') \quad (d(w_1) + d(w_2) \leq k) \\ &= \bar{f}(W) * \bar{f}(W'). \end{aligned}$$

Now we are left to show that, for any word $w \in \mathcal{A}$, we have

$$(30) \quad \bar{f}(P_X(w)) = Q(\bar{f}(w)).$$

Corresponding to the four components of the disjoint union decomposition in Eq. (22), we have the following four cases to consider.

If w is in $\widetilde{\mathcal{A}}^+$, then by definition $P_X(w) = \lfloor w \rfloor \in \mathcal{A}$. By definition (Eq. (28)) we have

$$\bar{f}(P_X(w)) = \bar{f}(\lfloor w \rfloor) = Q(\bar{f}(w)).$$

If w is in $\sqcup_{r \geq 1} (\mathcal{A}_0[\widetilde{\mathcal{A}}_{n-1}^+]^{r-1} \mathcal{A}_0[\widetilde{\mathcal{A}}_{n-2}^+]^{(2)})$, then write $w = w_1 \lfloor w_2 \rfloor^{(s)}$, $s \geq 2$ with $w_2 \in \mathcal{A}^+$. Then $P_X(w) = \lfloor w_1 \lfloor w_2 \rfloor \rfloor^{(s)}$. We have

$$\bar{f}(P_X(w)) = \bar{f}(\lfloor w_1 \lfloor w_2 \rfloor \rfloor^{(s)}) = Q^s(\bar{f}(w_1 \lfloor w_2 \rfloor)) = Q^s(\bar{f}(w_1) * Q(\bar{f}(\lfloor w_2 \rfloor))).$$

Since Q is an averaging operator, we further have

$$\bar{f}(P_X(w)) = Q(\bar{f}(w_1) * Q^s(\bar{f}(w_2))) = Q(\bar{f}(w_1) * \bar{f}(\lfloor w_2 \rfloor^{(s)})) = Q(\bar{f}(w)).$$

If w is in $\sqcup_{r \geq 1} (\lfloor \widetilde{\mathcal{A}}^+ \rfloor \mathcal{A}_0)^r$, then write $w = \lfloor w_1 \rfloor^{(s)} w_2$ with $w_1 \in \mathcal{A}^+$. Then by definition $P_X(w) = \lfloor w_1 \diamond \lfloor w_2 \rfloor \rfloor^{(s)}$. Thus

$$\bar{f}(P_X(w)) = \bar{f}(\lfloor w_1 \diamond \lfloor w_2 \rfloor \rfloor^{(s)}) = Q^s(\bar{f}(w_1 \diamond \lfloor w_2 \rfloor)) = Q^s(\bar{f}(w_1) * \bar{f}(\lfloor w_2 \rfloor)) = Q^s(\bar{f}(w_1) * Q(\bar{f}(w_2))).$$

Since Q is an averaging operator, we further have

$$\bar{f}(P_X(w)) = Q(Q^s(\bar{f}(w_1)) * \bar{f}(w_2)) = Q(\bar{f}(\lfloor w_1 \rfloor^{(s)} w_2)) = Q(\bar{f}(w)).$$

If w is in $\sqcup_{r \geq 1} (\lfloor \widetilde{\mathcal{A}}^+ \rfloor \mathcal{A}_0)^r \lfloor \widetilde{\mathcal{A}}^+ \rfloor$, then write $w = \lfloor w_1 \rfloor^{(s)} w_2 \lfloor w_3 \rfloor^{(t)}$ with $w_1, w_2, w_3 \in \mathcal{A}^+$. Then $P_X(w) = \lfloor w_1 \diamond \lfloor w_2 \lfloor w_3 \rfloor \rfloor \rfloor^{(s+t-1)}$. We have

$$\begin{aligned} \bar{f}(P_X(w)) &= \bar{f}(\lfloor w_1 \diamond \lfloor w_2 \lfloor w_3 \rfloor \rfloor \rfloor^{(s+t-1)}) \\ &= Q^{s+t-1}(\bar{f}(w_1 \diamond \lfloor w_2 \lfloor w_3 \rfloor \rfloor)) \\ &= Q^{s+t-1}(\bar{f}(w_1) * \bar{f}(\lfloor w_2 \lfloor w_3 \rfloor \rfloor)) \\ &= Q^{s+t-1}(\bar{f}(w_1) * Q(\bar{f}(w_2 \lfloor w_3 \rfloor))) \\ &= Q^s(\bar{f}(w_1) * Q^t(\bar{f}(w_2 \lfloor w_3 \rfloor))) \quad (Q \text{ is an averaging operator}) \\ &= Q^s(\bar{f}(w_1) * Q^t(\bar{f}(w_2) * Q(\bar{f}(w_3)))) \\ &= Q^s(\bar{f}(w_1) * Q(\bar{f}(w_2) * Q^t(\bar{f}(w_3)))) \quad (Q \text{ is an averaging operator}) \\ &= Q(Q^s(\bar{f}(w_1)) * (\bar{f}(w_2) * Q^t(\bar{f}(w_3)))) \quad (Q \text{ is an averaging operator}) \\ &= Q(\bar{f}(\lfloor w_1 \rfloor^{(s)}) * (\bar{f}(w_2) * \bar{f}(\lfloor w_3 \rfloor^{(t)}))) \\ &= Q(\bar{f}(w)). \end{aligned}$$

Therefore \bar{f} commutes with the averaging operator. This completes the proof of Theorem 2.12.(c). The proof of Theorem 2.12 is now completed.

3. ENUMERATION IN AVERAGING ALGEBRAS AND LARGE SCHRÖDER NUMBERS

In this section, we study the enumeration and generating functions of free averaging algebras. We first give the generating function of averaging words in two variables parameterizing the number of appearances of the variable and the operator respectively. We then observe that the generating function in one variable for averaging words (resp. indecomposable averaging words) with one idempotent operator and one idempotent generator is twice (resp. z times) the generating function of large Schröder numbers. This motivates us to give two interpretations of large Schröder numbers in terms of averaging words and a class of decorated rooted trees. As a result, we obtain a recursive formula for large Schröder numbers.

3.1. Enumeration of averaging words. In this section, we restrict ourselves to the set of averaging words with one generator and one idempotent operator and then give some results on enumerations of this set. For an idempotent operator, P is an averaging operator if and only if it is a Reynolds operator. Interestingly, in most applications of averaging algebras in physics (invariant theory and fluid dynamics), function spaces, Banach algebras, the operators are idempotent.

Under the condition that the operator $\lfloor \rfloor$ is idempotent, no two pairs of brackets can be immediately adjacent or nested in an averaging word. Enumerations of Rota-Baxter words are given in [15]. We will follow the similar notations and apply the similar method to solve the enumeration problem of averaging words.

For an averaging word w , an **x -run** is any occurrence in w of consecutive products of x of maximal length. Let v be either a positive integer or ∞ and let \mathfrak{A}_v be the subset of \mathcal{A} (including $\mathbf{1}$ ¹) where the length of x -runs is $\leq v$ with the convention that there is no restriction on x -runs when $v = \infty$. The number of balanced pairs of brackets (resp. of x) in an averaging word is called its **degree** (resp. **arity**). For $n \geq 1$, let $\mathfrak{A}_v(n)$ denote the subset of \mathfrak{A}_v consisting of all averaging words of degree n . For $m \geq 1$, let $\mathfrak{A}_v(n, m)$ denote the subset of \mathfrak{A}_v consisting of averaging words with degree n and arity m . Moreover, for $1 \leq k \leq m$, we let $\mathfrak{A}_v(n, m; k)$ be the subset of $\mathfrak{A}_v(n, m)$ consisting of averaging words where the m x 's are distributed into exactly k x -runs.

Let $G(m, k, v)$ be the set of compositions of the integer m into k positive integer parts, with each part at most v and let $g(m, k, v)$ be the size of this set ($v = \infty$ means there is no restrictions on the size of each part). Then we have [15, 21]

$$(31) \quad G_{k,v}(t) := \sum_{m=1}^{\infty} g(m, k, v) t^m = t^k \left(\frac{1 - t^v}{1 - t} \right)^k$$

and

$$(32) \quad G_{k,\infty}(t) := \sum_{m=1}^{\infty} g(m, k, \infty) t^m = \left(\frac{t}{1 - t} \right)^k.$$

In particular, by Eq. (31), we have $v = 1$ implies $G_{k,1}(t) = t^k$.

By the definition of $\mathfrak{A}_v(n, m)$, we have the disjoint union

$$(33) \quad \mathfrak{A}_v(n, m) = \bigsqcup_{k=1}^m \mathfrak{A}_v(n, m; k).$$

By the definition of $\mathfrak{A}_1(n, m; k)$, we have that $\mathfrak{A}_1(n, m; k) \neq \emptyset$ implies $m = k$. Then we have $\mathfrak{A}_1(n, m; k) = \mathfrak{A}_1(n, k)$. We define a map

$$(34) \quad \Phi_{v,n,m} : \mathfrak{A}_v(n, m; k) \longrightarrow \mathfrak{A}_1(n, k)$$

by sending $w \in \mathfrak{A}_v(n, m; k)$ to the averaging word $\Phi_{v,n,m}(w)$ in $\mathfrak{A}_1(n, k)$ obtained by replacing each of the x -runs appearing in w by a single x . This map is clearly surjective for each pair (n, k) . Further, each fiber (inverse image) of $\Phi_{v,n,m}$ has $g(m, k, v)$ elements, giving rise to a bijection

$$(35) \quad \Psi : \mathfrak{A}_v(n, m; k) \longleftrightarrow \mathfrak{A}_1(n, k) \times G(m, k, v).$$

Therefore by Eq. (33), the numbers $a_v(n, m)$ of averaging words of degree n and arity m in the set \mathfrak{A}_v are given by

$$a_v(n, m) = \sum_{k=1}^m g(m, k, v) a_1(n, k).$$

¹In order to consider x to be an associate in Section 3.2, we add the trivial averaging word $\mathbf{1}$ to \mathcal{A} .

We next determine the expressions of the generating functions $\mathfrak{A}_v(z, t)$ of the number sequences $\alpha_v(n, m)$, $n, m \geq 0$, for $1 \leq v \leq \infty$.

Theorem 3.1. *Let $1 \leq v \leq \infty$. The generating function $\mathfrak{A}_v(z, t)$ for the number $\alpha_v(n, m)$, $n, m \geq 0$, of averaging words is given by $\mathfrak{A}_v(z, t) = \mathfrak{A}_1(z, G_{1,v}(t))$. where $G_{1,v}$ is given by Eq. (31) for finite v and by Eq. (32) for infinite v .*

Proof. We have

$$\begin{aligned}
 \mathfrak{A}_v(z, t) : &= \sum_{n, m \geq 1} \alpha_v(n, m) z^n t^m \\
 &= \sum_{n, m \geq 1} \sum_{k=1}^m g(m, k, v) \alpha_1(n, k) z^n t^m \\
 &= \sum_{m \geq 1} \left(\sum_{k \geq 1} g(m, k, v) \left(\sum_{n \geq 1} \alpha_1(n, k) z^n \right) \right) t^m \\
 &= \sum_{k \geq 1} \left(\sum_{n \geq 1} \alpha_1(n, k) z^n \right) (G_{1,v}(t))^k \\
 &= \mathfrak{A}_1(z, G_{1,v}(t)).
 \end{aligned}$$

□

By Eq. (32) we also have

Corollary 3.2. We have the generating function

$$\mathfrak{A}_\infty(z, t) = \mathfrak{A}_1\left(z, \frac{t}{1-t}\right).$$

Now we have reduced the problem of finding the explicit expression of the generating function for $\alpha_\infty(n, m)$ to the problem of finding $\mathfrak{A}_1(z, t)$, to be considered in the next subsection.

3.2. The generating function $\mathfrak{A}_1(z, t)$. In this section, we will focus on the generating function $\mathfrak{A}_1(z, t)$ for $\alpha_1(n, m)$. First, we give some descriptions of the word structures of averaging words. Note that \mathfrak{A}_1 is the subset consisting of **1** and averaging words w composed of x 's and pairs of balanced brackets such that no two x 's are adjacent, and no two pairs of brackets can be immediately adjacent or nested. This special case can be considered as the case where we assume $X = \{x\}$, $x^2 = x$ and $\lfloor \rfloor^2 = \lfloor \rfloor$.

In the rest of this section, all averaging words are assumed to be in \mathfrak{A}_1 .

For $n > 0$, let $B(n)$ be the subset of $\mathfrak{A}_1(n)$ consisting of averaging words that begin with a left bracket and end with a right bracket and words in $B(n)$ are said to be **bracketed**. By pre- or post-concatenating a bracketed averaging word w with x , we get three new averaging words: xw , wx , and xwx , which are called respectively the **left**, **right** and **bilateral associate** of w . Moreover, left, right and bilateral associate of w are collectively referred to as the **associates** of w . We also consider x to be an associate of the trivial word **1**. Any nontrivial averaging word is either bracketed or an associate. Thus for $n > 0$, the set $C(n)$ of all associates is the disjoint union

$$C(n) = xB(n) \sqcup B(n)x \sqcup xB(n)x, \quad n > 0.$$

and forms the complement of $B(n)$ in $\mathfrak{A}_1(n)$. Define $C^-(n) := B(n)x$. We call the words in the subset $xB(n) \sqcup xB(n)x$ **admissible**. The set of bracketed averaging words is further divided into

two disjoint subsets. The first subset $I(n)$ consists of all indecomposable bracketed averaging words whose beginning left bracket and ending right bracket are paired, like $[x[x]]$. The second subset $D(n)$ consists of all decomposable bracketed averaging words whose beginning left bracket and ending right bracket are not paired, like $[x]x[x]$. For the convenience in counting, we define $B(0), I(0), D(0)$ to be the empty set and note that $C(0) = \{x\}$. With our convention, denote $\mathfrak{A}(0) := \mathfrak{A}_1(0) = \{\mathbf{1}, x\}$.

The following table lists these various types of averaging words in lower degrees

deg	I(n)	D(n)	C(n)	$C^-(n)$	B(n)
0			x		
1	$[x]$		$x[x], [x]x, x[x]x$	$[x]x$	$[x]$
2	$[x[x]], [x[x]x]$	$[x]x[x]$	9 associates	$B(2)x$	$I(2) \cup D(2)$

The production rules will be

- $$\begin{aligned}
 (36) \quad \langle AW \rangle &\longrightarrow \mathbf{1} \mid \langle bracketed \rangle \mid \langle associate \rangle \\
 (37) \quad \langle associate \rangle &\longrightarrow x \mid x\langle bracketed \rangle \mid \langle bracketed \rangle x \mid x\langle bracketed \rangle x \\
 (38) \quad \langle bracketed \rangle &\longrightarrow \langle indecomposable \rangle \mid \langle decomposable \rangle \\
 (39) \quad \langle indecomposable \rangle &\longrightarrow [\langle admissible \rangle] \\
 (40) \quad \langle decomposable \rangle &\longrightarrow \langle bracketed \rangle x \langle bracketed \rangle
 \end{aligned}$$

An averaging word w has arity m means the number of occurrences of x in w is m . For any $m \geq 0$, let $\mathfrak{A}(n, m)$ be the subset of \mathfrak{A} consisting of words with degree n and arity m , and define similarly the notations $C(n, m)$, $C^-(n, m)$, $B(n, m)$, $I(n, m)$, and $D(n, m)$. These are all finite sets. Let their sizes be $a_{n,m}$, $c_{n,m}$, $c_{n,m}^-$, $b_{n,m}$, $i_{n,m}$, and $d_{n,m}$ respectively. For initial values, we have

$$\begin{aligned}
 a_{0,0} &= 1; & c_{0,0} &= b_{0,0} = i_{0,0} = d_{0,0} = 0; \\
 a_{0,1} &= c_{0,1} = 1; & b_{0,1} &= i_{0,1} = d_{0,1} = 0; \\
 a_{1,1} &= b_{1,1} = i_{1,1} = 1; & c_{1,1} &= d_{1,1} = 0; \\
 a_{1,2} &= c_{1,2} = 2; & b_{1,2} &= i_{1,2} = d_{1,2} = 0; \\
 a_{1,3} &= c_{1,3} = 1; & b_{1,3} &= i_{1,3} = d_{1,3} = 0;
 \end{aligned}$$

$$\begin{aligned}
 a_{0,m} &= c_{0,m} = b_{0,m} = i_{0,m} = d_{0,m} = 0 && \text{for } m \geq 2; \\
 a_{1,m} &= c_{1,m} = b_{1,m} = i_{1,m} = d_{1,m} = 0 && \text{for } m \geq 4; \\
 a_{n,0} &= c_{n,0} = b_{n,0} = i_{n,0} = d_{n,0} = 0 && \text{for } n \geq 1; \\
 a_{n,1} &= c_{n,1} = b_{n,1} = i_{n,1} = d_{n,1} = 0 && \text{for } n \geq 2.
 \end{aligned}$$

From the production rules (36)–(40), we see that for $n \geq 1, m \geq 2$,

- $$\begin{aligned}
 (41) \quad a_{n,m} &= b_{n,m} + c_{n,m}, \\
 (42) \quad c_{n,m} &= 2b_{n,m-1} + b_{n,m-2}, \\
 (43) \quad b_{n,m} &= i_{n,m} + d_{n,m}, \\
 (44) \quad i_{n,m} &= c_{n-1,m} - c_{n-1,m}^- = c_{n-1,m} - b_{n-1,m-1},
 \end{aligned}$$

where Eq. (44) follows from Eq. (39) and $C^-(n) = B(n)x$.

Now for $n \geq 2$, $m \geq 2$ and $w \in D(n, m)$, we can write w uniquely as $w_{n_1} x w_{n_2} \cdots x w_{n_p}$ where $w_{n_j} \in I(n_j)$ and $n_1 + \cdots + n_p$ is a composition of n using p positive integers. Let m_j be the arity of w_{n_j} . Then clearly, $m_1 + \cdots + m_p = m - p + 1$. So we have

$$(45) \quad d_{n,m} = \sum_{p=2}^{\min(n,m)} \sum_{(m_1, \dots, m_p; m-p+1)} \sum_{(n_1, \dots, n_p; n)} (i_{n_1, m_1}) \cdots (i_{n_p, m_p}).$$

The case when $p = 1$ corresponds to a single summand $i_{n,m}$, and then

$$(46) \quad b_{n,m} = i_{n,m} + d_{n,m} = \sum_{p=1}^{\min(n,m)} \sum_{(m_1, \dots, m_p; m-p+1)} \sum_{(n_1, \dots, n_p; n)} (i_{n_1, m_1}) \cdots (i_{n_p, m_p}).$$

Now from Eqs.(42) and (44), we have

$$(47) \quad i_{n,m} = c_{n-1,m} - b_{n-1,m-1} = 2b_{n-1,m-1} + b_{n-1,m-2} - b_{n-1,m-1} = b_{n-1,m-1} + b_{n-1,m-2}.$$

Define the bivariate generating series

$$\mathfrak{A}(z, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} z^n t^m$$

and similarly define $B(z, t)$, $I(z, t)$, $D(z, t)$, and $C(z, t)$. Note that for $B(z, t)$, $I(z, t)$ and $D(z, t)$, it does not matter whether the series indices n, m start at 0 or 1. We will multiply both sides of Eq.(47) by $z^n t^m$ and sum up for $n \geq 2$, $m \geq 2$. Since $i_{0,m} = i_{n,0} = 0$ for all $m, n \geq 0$ and $i_{1,m} = i_{n,1} = 0$ for all $m, n \geq 2$, then the left hand side gives

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} i_{n,m} z^n t^m = I(z, t) - zt.$$

Now, we sum up the right hand side of Eq.(47) one term at a time.

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} b_{n-1,m-1} z^n t^m &= zt \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} z^n t^m = ztB(z, t), \\ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} b_{n-1,m-2} z^n t^m &= zt^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{n,m} z^n t^m = zt^2(B(z, t) + \sum_{n=1}^{\infty} b_{n,0} z^n) = zt^2 B(z, t). \end{aligned}$$

Hence, we have the identity

$$(48) \quad I(z, t) - zt = zt(1 + t)B(z, t).$$

Using Eq. (46), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} z^n t^m &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\min(m,n)} \sum_{(m_1, \dots, m_p; m-p+1)} \sum_{(n_1, \dots, n_p; n)} (i_{n_1, m_1}) \cdots (i_{n_p, m_p}) z^n t^m \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\min(m,n)} \sum_{(m_1, \dots, m_p; m-p+1)} \sum_{(n_1, \dots, n_p; n)} (i_{n_1, m_1} z^{n_1} t^{m_1}) \cdots (i_{n_p, m_p} z^{n_p} t^{m_p}) z^n t^m \\ &= \sum_{p=1}^{\infty} \left(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} i_{k,\ell} z^k t^\ell \right)^p t^{p-1} \end{aligned}$$

$$= \sum_{p=1}^{\infty} I(z, t)^p t^{p-1}$$

and hence

$$(49) \quad B(z, t) = \frac{I(z, t)}{1 - tI(z, t)}.$$

Thus we obtained the identity defining $I(z, t)$ as

$$(50) \quad I(z, t) - zt = zt(1 + t) \frac{I(z, t)}{1 - tI(z, t)}.$$

Solving this quadratic equation in $I(z, t)$ and using the initial conditions, we find

$$(51) \quad I(z, t) = \frac{1 - zt - \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2t}.$$

Then by Eq. (49), we have

$$(52) \quad B(z, t) = \frac{1 - zt - 2zt^2 - \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2zt^2(1 + t)}.$$

Furthermore, from Eq. (46), we obtain

$$(53) \quad \begin{aligned} D(z, t) &= B(z, t) - I(z, t) \\ &= \frac{1 - 2zt - 3zt^2 + z^2 t^2 + z^2 t^3 + (zt + zt^2 - 1) \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2zt^2(1 + t)}. \end{aligned}$$

We can also obtain the bivariate generating series for $c_{n,m}$:

$$\begin{aligned} C(z, t) &= \sum_{n=0}^{\infty} (c_{n,0} z^n + c_{n,1} z^n t + \sum_{m=2}^{\infty} c_{n,m} z^n t^m) \\ &= t + \sum_{m=2}^{\infty} c_{0,m} t^m + \sum_{n=1}^{\infty} \left(\sum_{m=2}^{\infty} c_{n,m} z^n t^m \right) \\ &= t + \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} (2b_{n,m-1} + b_{n,m-2}) z^n t^m \\ &= t + \sum_{n=1}^{\infty} \left(t \sum_{m=1}^{\infty} 2b_{n,m} z^n t^m \right) + t^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{n,m} z^n t^m \\ &= t + 2tB(z, t) + t^2 B(z, t), \end{aligned}$$

giving

$$(54) \quad A(z, t) = \frac{2 + t - 2zt - 3zt^2 - (2 + t) \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2zt(1 + t)}$$

by Eq. (52).

Finally, utilizing $\mathfrak{A}(z, t) = 1 + B(z, t) + C(z, t)$, Eqs.(52) and (54) we derive the following generating function $\mathfrak{A}_1(z, t) = \mathfrak{A}(z, t)$ for the sequences $a_{n,m} = a_1(n, m)$.

$$(55) \quad \mathfrak{A}(z, t) = \frac{(1 + t)(1 - zt - \sqrt{z^2 t^2 - (2t + 4t^2)z + 1})}{2zt^2}.$$

Recall that the averaging words in \mathfrak{A}_1 can be considered as the case where we take $X = \{x\}$, $x^2 = x$ and $\lfloor \rfloor^2 = \lfloor \rfloor$. We conclude our discussion on generating functions by ignoring the arity and only focus on the degree. For $n > 0$, let a_n (resp. c_n , resp. b_n , resp. i_n , resp. d_n) be the number of all (resp. associate, resp. bracketed, resp. indecomposable, resp. decomposable) averaging words with n pairs of (balanced) brackets. For example, we have $a_0 = 2$ since such averaging words with no operators are $\{1, x\}$; while $a_1 = 4$ since those with one operators are $\{ \lfloor x \rfloor, x \lfloor x \rfloor, \lfloor x \rfloor x, x \lfloor x \rfloor x \}$.

Putting $t = 1$ in the generating functions of $\mathfrak{A}(z, t)$, $B(z, t)$, $I(z, t)$, $D(z, t)$ and $C(z, t)$, we obtain

Theorem 3.3. *The generating series for a_n , b_n , i_n , d_n and c_n are given by*

$$\begin{aligned}
 \mathfrak{A}(z) &= \sum_{n=0}^{\infty} a_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{z}, \\
 B(z) &= \sum_{n=0}^{\infty} b_n z^n = \frac{1 - 3z - \sqrt{z^2 - 6z + 1}}{4z}, \\
 I(z) &= \sum_{n=0}^{\infty} i_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{2}, \\
 D(z) &= \sum_{n=0}^{\infty} d_n z^n = \frac{1 - 5z + 2z^2 + (2z - 1)\sqrt{z^2 - 6z + 1}}{4z}, \\
 C(z) &= \sum_{n=0}^{\infty} c_n z^n = \frac{3 - 5z - 3\sqrt{z^2 - 6z + 1}}{4z}.
 \end{aligned}
 \tag{56}$$

From the theorem we obtain

Corollary 3.4. (a) The sequence $a_n, n \geq 0$, of averaging words of degree n with one idempotent generator and one idempotent operator, is twice the sequence s_n of large Schröder numbers: $a_n/2 = s_n, n \geq 0$. The first few terms of $a_n, n \geq 0$, are

$$2, 4, 12, 44, 180, 788, 3612, 17116, \dots$$

(b) The sequence $i_n, n \geq 1$, of bracketed indecomposable averaging words of degree n is the sequence $s_n, n \geq 0$, of large Schröder numbers: $i_{n+1} = s_n, n \geq 0$. The first few terms of $i_n, n \geq 0$, are

$$0, 1, 2, 6, 22, 90, 394, 1806, \dots$$

Proof. Both results are proved by comparing the corresponding generating functions with the generating function

$$S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{2z}$$

of large Schröder sequence (A006318 in the On-line Encyclopedia of Integer Sequences [28]).

□

On the other hand, the number d_n of bracketed decomposable averaging words of degree n ($n \geq 0$) is a sequence which starts with

$$0, 0, 1, 5, 23, 107, 509, 2473, \dots$$

This sequences is new and can not be found in [28].

3.3. Averaging words and large Schröder numbers. The sequence $s_n, n \geq 0$, of large Schröder numbers (A006318 in [28]) is an important sequence of integers with numerous interesting properties and interpretations. For example, s_n counts the number of Schröder paths of semilength n , namely lattice paths on the plane from $(0, 0)$ to $(2n, 0)$ that do not go below the x -axis and consist of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (2, 0)$. See [28] for more details.

Corollary 3.4 gives two more interpretations of large Schröder numbers in terms of averaging words with a single idempotent generators and idempotent operator. Motivated by this, we next give another interpretation of the sequence of large Schröder numbers in terms of decorated trees.

Definition 3.5. (a) Let \mathfrak{T} denote the set of planar reduced rooted trees together with the trivial tree $|$. Let ω and ι be symbols. By a (ω, ι) -**decorated tree** we mean a tree t in \mathfrak{T} together with a decoration on the vertices of t by ω and a decoration on the leaves of t by either ω or ι . Let $\mathcal{D}(t)$ denote the set of (ω, ι) -decorated trees from t and denote

$$\mathcal{D}(\mathfrak{T}) := \bigsqcup_{t \in \mathfrak{T}} \mathcal{D}(t).$$

- (b) Let t be in \mathfrak{T} whose number of leaves is greater than 1. Then there exists an integer m such that t can be written uniquely as the **grafting** $\bigvee_{i=1}^m t_i := t_1 \vee \cdots \vee t_m$ of trees t_1, t_2, \dots, t_m . The trees t_1, \dots, t_m are called the **branches** of t .
- (c) Let $\tau \in \mathcal{D}(t)$ where t is in \mathfrak{T} with grafting $t = t_1 \vee \cdots \vee t_m$. Then there are $\tau_i \in \mathcal{D}(t_i), 1 \leq i \leq m$, such that τ is the grafting $\omega(\bigvee_{i=1}^m \tau_i) = \omega(\tau_1 \vee \cdots \vee \tau_m)$ of τ_1, \dots, τ_m with the new root decorated by ω .

Since the rooted trees we are considering are reduced, we have $m \geq 2$ in any grafting of trees. Now, we define a special subset of $\mathcal{D}(\mathfrak{T})$.

Definition 3.6. (a) A (ω, ι) -decorated tree $\tau \in \mathcal{D}(\mathfrak{T})$ is called a **Schröder tree** if either τ is the trivial tree decorated by ω , or τ is a tree with more than one leaves and satisfies the following conditions: For each vertex v of τ , let τ_v be the subtree of τ with root v . Then

- (i) the leftmost branch of τ_v is a leaf decorated by ι ;
- (ii) the branches of τ_v are alternatively a leaf decorated by ι and a subtree that is not a leaf decorated by ι (the latter means that the subtree is either not a leaf or a leaf decorated by ω).

To put it in another way, let τ_v be of the form $\omega(\tau_{v,1} \vee \cdots \vee \tau_{v,k})$, then each $\tau_{v,i}$ for i odd is a leaf decorated by ι and each $\tau_{v,i}$ for i even is either not a leaf or is a leaf decorated by ω .

- (b) Let ST denote the set of Schröder trees and let ST_n denote the subset of ST consisting of those Schröder trees whose n vertices or leaves are decorated by ω .

Theorem 3.7. (a) The sequence $\{|\text{ST}_n|\}_{n \geq 1}$ counting Schröder trees is the sequence $\{i_n\}_{n \geq 1}$ counting indecomposable averaging words: $|\text{ST}_n| = i_n, n \geq 1$.

(b) The sequence $\{|\text{ST}_n|\}_{n \geq 1}$ coincides the sequence $\{s_n\}_{n \geq 0}$ of large Schröder numbers. In other words,

$$|\text{ST}_{n+1}| = i_{n+1} = s_n, n \geq 0.$$

- (c) The sequence $\{s_n\}_{n \geq 0}$ of large Schröder numbers satisfies the following recursion:

$$s_0 = 1,$$

$$(57) \quad s_n = 2 \sum_{j=1}^n \sum_{(p_1, \dots, p_j) \in G(n, j)} s_{p_1-1} \cdots s_{p_j-1},$$

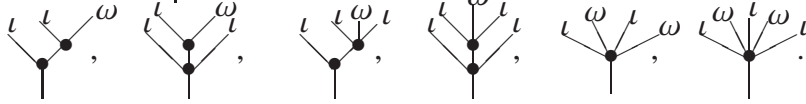
where $G(n, j)$ is the set of compositions of the integer n into j positive integer parts.

We illustrate the theorem by listing the first three terms of the sequences $I(n)$ and ST_n , $n \geq 1$.

	I(1)	$[x]$	1
1.	I(2)	$[x[x]], [x[x]x]$	2
	I(3)	$[x[x[x]]], [x[x[x]x]], [x[x[x]x]x], [x[x[x]x]x]x, [x[x]x[x]x], [x[x]x[x]x]x]$	6

2. $ST_1 : \begin{array}{c} \omega \\ | \end{array}$;

$ST_2 :$ 

$ST_3 :$ 

Proof. (a) We just need to prove that the sequence $\{|ST_n|\}_{n \geq 1}$ and $\{i_n\}_{n \geq 1}$ satisfy the same recursion relation and the same initial condition.

Let $I(n)$ denote the set of bracketed indecomposable averaging words of degree n and I by the set of all bracketed indecomposable averaging words.

First we have $I(1) = \{[x]\}$ and $ST_1 = \{\begin{array}{c} \omega \\ | \end{array}\}$. Hence $|ST_1| = i_1 = 1$.

Next, let $n \geq 2$. For any word $W = [w_1 w_2 \cdots w_m] \in I(n)$, we have $w_i = x$ for i odd and $w_i \in I$ for i even. Then there exist p_i , $1 \leq i \leq k := \lfloor m/2 \rfloor$ such that w_{2i} is in $I(p_i)$. Thus we have $p_1 + \cdots + p_k = n - 1$ and so $(p_1, \dots, p_k) \in G(n - 1, k)$. Note that there are two W 's that give the same (p_1, \dots, p_k) : one is $[w_1 \cdots w_{2k}]$, the another is $[w_1 \cdots w_{2k}x]$.

Conversely, let $(p_1, \dots, p_k) \in G(n - 1, k)$ and take $w_2 \in I(p_1), \dots, w_{2k} \in I(p_k)$ and $w_1 = \cdots = w_{2k-1} = w_{2k+1} = x$. Then $[w_1 \cdots w_{2k}]$ and $[w_1 \cdots w_{2k+1}]$ are in $I(n)$.

Therefore, we have the following recursive formula for i_n , $n \geq 2$:

$$(58) \quad i_n = 2 \left(i_{n-1} + \cdots + \sum_{(p_1, \dots, p_j) \in G(n-1, j)} i_{p_1} \cdots i_{p_j} + \cdots + i_1^{n-1} \right) = 2 \sum_{j=0}^{n-1} \sum_{(p_1, \dots, p_j) \in G(n-1, j)} i_{p_1} \cdots i_{p_j}.$$

On the other hand, if $\tau \in ST_n$, then there exists $k \geq 0$ such that either $\tau = \omega(\bigvee_{i=1}^{2k} \tau_i)$ or $\omega(\bigvee_{i=1}^{2k+1} \tau_i)$. Furthermore, each τ_i with i odd is a leaf decorated by ι and each τ_{2i} , $1 \leq i \leq k$, is in ST_{p_i} for some integer $p_i \geq 1$. Since ω does not appear in τ_i for i odd, we have $p_1 + p_2 + \cdots + p_k = n - 1$. That is (p_1, \dots, p_k) is in $G(n - 1, k)$.

Conversely, let (p_1, \dots, p_k) be in $G(n - 1, k)$. Take $\tau_{2i} \in ST_{p_i}$, $1 \leq i \leq k$ and take τ_i for odd i to be a leaf decorated by ι . Then the (ω, ι) -decorated trees $\omega(\bigvee_{i=1}^{2k} \tau_i)$ and $\omega(\bigvee_{i=1}^{2k+1} \tau_i)$ are in ST_n .

By the above argument, we obtain the following recursive formula for $|ST_n|$, $n \geq 2$.

$$(59) \quad \begin{aligned} |ST_n| &= 2 \left(|ST_{n-1}| + \cdots + \sum_{(p_1, \dots, p_j) \in G(n-1, j)} (|ST_{p_1}| \cdots |ST_{p_j}|) + \cdots + |ST_1|^{n-1} \right) \\ &= \sum_{j=0}^{n-1} \sum_{(p_1, \dots, p_j) \in G(n-1, j)} (|ST_{p_1}| \cdots |ST_{p_j}|). \end{aligned}$$

In summary, $i_n, n \geq 1$ and $|\text{ST}_n|, n \geq 1$ have the same initial value and the same recursive relation. Therefore $i_n = |\text{ST}_n|, n \geq 1$.

(b) This follows from Item (a) and Corollary 3.4.

(c) This follows from Item (b) and Eq. (59). \square

Remark 3.8. As pointed out by the referee, the recursive formula in Eq. (57) also follows from the classical identity

$$zS(z)^2 + (z-1)S(z) + 1 = 0$$

of the generating function of large Schröder sequence, since the identity can be reformulated as

$$S(z) = 1 + \frac{2zS(z)}{1 - zS(z)},$$

which, when developed, gives Eq. (57). The proof we provided reveals better the combinatorics behind the recursion. This will be useful for example in the following remark.

Remark 3.9. Theorem 3.7.(a) suggests that, for each $n \geq 1$, ST_n is in bijection with the set of indecomposable averaging words $\text{I}(n)$ of degree n , and hence with the set of admissible averaging words of degree n thanks to Eq. (39). We show that this is indeed the case by inductively constructing bijections $\Psi_n : \text{I}(n) \rightarrow \text{ST}_n, n \geq 1$, using the proof of Theorem 3.7.(a).

First, define $\Psi_1(\lfloor x \rfloor) = \omega$. Assume that Ψ_k has been defined for $1 \leq k \leq n$. As in the proof of Theorem 3.7.(a), any word $W \in \text{I}(n+1)$ is of the form $W = \lfloor w_1 w_2 \cdots w_m \rfloor$, where $w_i = x$ for i odd and $w_i \in \text{I}(p_{2/i})$ for i even, with $1 \leq p_{i/2} \leq n$ and $p_1 + \cdots + p_{\lfloor m/2 \rfloor} = n$. Then define

$$\Psi_{n+1}(W) = \omega \left(\bigvee_{i=1}^m \Psi'(w_i) \right) \quad \text{where} \quad \Psi'(w_i) = \begin{cases} \omega & i \text{ is odd,} \\ \Psi_{p_{2/i}}(w_i) & i \text{ is even.} \end{cases}$$

For example, we have

$$\Psi_2(\lfloor x \rfloor) = \omega, \quad \Psi_2(\lfloor x \rfloor \lfloor x \rfloor) = \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}, \quad \Psi_3(\lfloor x \rfloor \lfloor x \rfloor \lfloor x \rfloor) = \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}, \quad \Psi_3(\lfloor x \rfloor \lfloor x \rfloor \lfloor x \rfloor \lfloor x \rfloor) = \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}.$$

4. TREE REPRESENTATION AND OPERAD OF AVERAGING ALGEBRAS

In this section, we identify the set $\mathcal{A}(\{x\})$ of averaging words on a singleton $\{x\}$ with a class of unreduced binary trees. This identification gives a combinatorial description of $\mathcal{A}(\{x\})$ and hence the free averaging algebra on $\{x\}$, without the idempotency assumption in the previous section. This identification also allows us to construct the operad of averaging algebras.

4.1. Unreduced binary trees and averaging words on a singleton. Recall that an unreduced binary tree is a tree in which each vertex has either one input or two inputs. By convention the trivial tree is an unreduced binary tree. We let \mathcal{T} denote the set of unreduced binary trees. A vertex v of $\tau \in \mathcal{T}$ is called a **uni-vertex** (resp. **bi-vertex**) of τ if v is a vertex with one input (resp. two inputs). If v is a bi-vertex of τ , then τ has a left subtree and a right subtree associated to v , denoted by $\tau_{v,l}$ and $\tau_{v,r}$ respectively. For any $\tau \in \mathcal{T}$, denote the number of its leaves by lef_τ .

Denote the binary tree with two leaves by $t_\mu := \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}$, the tree with one leaf and one uni-vertex by $t_p := \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$, the trivial tree by $\text{id} := \begin{array}{c} \bullet \end{array}$. We use $t_\mu \circ (\tau_l \otimes \tau_r)$ to denote the tree obtained by grafting a tree τ_l on the left leaf of t_μ and a tree τ_r on the right leaf of t_μ ; $t_p \circ \tau$ to denote the tree obtained by grafting a tree τ on the only leaf of t_p . A **ladder** is a tree of the form $t_p^s, s \geq 1$. For example, we have

$$t_\mu \circ (\text{id} \otimes t_p) = \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}, \quad t_p \circ t_\mu = \begin{array}{c} \omega \\ \swarrow \quad \searrow \\ \bullet \end{array}, \quad t_p^2 = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

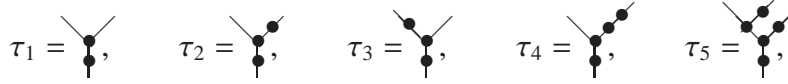
With these notations, any unreduced binary tree τ with $\text{lef}_\tau = 1$ (resp. $\text{lef}_\tau \geq 2$) can be uniquely expressed as $\tau = t_p^s$ (resp. $\tau = t_p^s \circ t_\mu \circ (\tau_\ell \otimes \tau_r)$) with $s \geq 0$ and τ_ℓ, τ_r unreduced binary trees. We use the convention that $t_p^0 = \text{id}$. The number $\text{pb}(\tau) := s$ is called the **bracketed power** of τ . An unreduced binary tree τ is called a **bracketed tree** (resp. **non-bracketed tree**) if $\text{pb}(\tau) > 0$ (resp. $\text{pb}(\tau) = 0$). For example, $t_p \circ t_\mu$ and t_p^2 are bracketed trees, while $t_\mu \circ (\text{id} \otimes t_p)$ is not. The trivial tree is non-bracketed. An unreduced binary tree with $\text{lef}_\tau \geq 2$ is non-bracketed if and only if its root is a bi-vertex.

We now define some special subsets of unreduced binary trees.

An unreduced binary tree τ is called a **left tree** if

- (a) for each bi-vertex v of τ , its right subtree $\tau_{v,r}$ is a leaf;
- (b) among the tree τ and the left subtrees of all the bi-vertices of τ , at most one is bracketed.

For example, in the list of trees



τ_1 is a left tree, while τ_2, τ_3, τ_4 and τ_5 are not.

For any $\tau \in \mathcal{T}$, let $\text{Lf}(\tau)$ denote the tree after replacing the right subtree of each bi-vertex of τ by a leaf. Then $\text{Lf}(\tau)$ automatically satisfies condition (a) for a left tree. By definition, for the trees τ_1, \dots, τ_5 above, we have

$$\text{Lf}(\tau_1) = \tau_1, \quad \text{Lf}(\tau_2) = \tau_1, \quad \text{Lf}(\tau_3) = \tau_3, \quad \text{Lf}(\tau_4) = \tau_1, \quad \text{Lf}(\tau_5) = \tau_1.$$

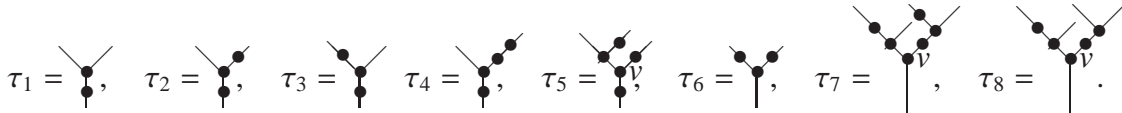
Let $\tau = t_p^s \circ t_\mu \circ (\tau_\ell \otimes \tau_r)$ be a bracketed tree with $\text{lef}_\tau \geq 2$. τ is called a **fundamental averaging tree** if $\text{Lf}(\tau)$ is a left tree and $\text{pb}(\tau_r) \leq 1$. For example, $\text{Lf}(\tau_1), \text{Lf}(\tau_2), \text{Lf}(\tau_4)$ and $\text{Lf}(\tau_5)$ are left trees, while $\text{Lf}(\tau_3)$ is not. It follows from $\text{pb}((\tau_4)_r) = 2$ that τ_4 is not a fundamental averaging tree. Hence, τ_1, τ_2 and τ_5 are fundamental averaging trees, while τ_3 and τ_4 are not.

The set Avt of **averaging trees** consists of all unreduced binary trees satisfying the following conditions:

- (a) Each bracketed subtree of τ is either a ladder or a fundamental averaging tree;
- (b) For each bi-vertex v of τ , $\tau_{v,r}$ is either trivial or bracketed such that either $\tau_{v,l}$ is trivial or has a trivial right subtree.

Remark 4.1. It follows from the definition of an averaging tree that a subtree of an averaging tree is still an averaging tree. The trivial tree and all ladders are averaging trees.

For example, consider the following eight unreduced binary trees



By the above rules, τ_1 and τ_2 are in Avt . τ_3 and τ_4 are bracketed subtrees of themselves respectively, but they are either ladders nor fundamental averaging trees, so τ_3 and τ_4 are not in Avt . $\text{Lf}(\tau_5)$ is a fundamental averaging tree. $\tau_{5v,r}$ is a ladder but the right subtree of $\tau_{5v,l}$ is not trivial. Hence τ_5 is not in Avt . Each bracketed subtree of τ_6 is a ladder, but both $\tau_{6v,r}$ and $\tau_{6v,l}$ are bracketed. Then $\tau_{6v,l}$ is not trivial or has a trivial right subtree. Thus τ_6 is not in Avt . Since $\tau_{7v,r} = \tau_3$ is not a fundamental averaging tree, the bracketed subtree $\tau_{7v,r}$ of τ_7 is not a fundamental averaging tree. Hence τ_7 is not in Avt . There are two bracketed subtrees of τ_8 , one is a ladder, the other is equal to τ_1 . Since $\tau_{v,l}$ has a trivial right subtree, τ_8 is in Avt .

There is a one to one correspondence between $AW(\{x\})$ and Avt . Recall that the number of x 's appearing in an averaging words W is called its arity and denote it by ar_W . We recursively define a map

$$\varphi : AW(\{x\}) \longrightarrow \mathcal{T}.$$

We do this first by letting $\varphi(x) = \text{id}$ and $\varphi(\lfloor x \rfloor^s) = t_p^s$. Assume that φ has been defined for any word W with $\text{ar}_W \leq n$, $n \geq 1$. For any word W with $\text{ar}_W = n + 1$, consider its standard decomposition $W = w_1 w_2 \cdots w_m \in AW$, where $w_i = x$ or $\lfloor V_i \rfloor^{s_i}$ with $V_i \in \Omega^+$. When $m \geq 2$, define

$$(60) \quad \varphi(W) = t_\mu \circ (\varphi(w_1 \cdots w_{m-1}) \otimes \varphi(w_m)).$$

When $m = 1$, W is the form of $\lfloor V \rfloor^s$ with $V \in \Omega^+$, then V is a word with $\text{ar}_V = n + 1$ and its breadth is ≥ 2 . Define

$$\varphi(W) = t_p^s \circ \varphi(V).$$

Proposition 4.2. *The map φ is a bijection from $\mathcal{A}(\{x\})$ to Avt .*

Proof. We divide the proof into three steps. First, we show the image of φ is a subset of Avt . Then we define a map $\Phi : \text{Avt} \longrightarrow AW(\{x\})$ such that $\varphi\Phi = \text{Id}_{\text{Avt}}$. Finally, we prove the map Φ also satisfies $\Phi\varphi = \text{Id}_{AW(\{x\})}$.

Step 1. $\varphi(\mathcal{A}(\{x\})) \subseteq \text{Avt}$: We prove it by induction on ar_W . When $\text{ar}_W = 1$, that is $W = x$ or $\lfloor x \rfloor^s$. Then $\varphi(W)$ is a trivial tree or ladder. Hence, $\varphi(W) \in \text{Avt}$.

Inductively assuming that $\varphi(W) \in \text{Avt}$ holds for any W with $\text{ar}_W \leq n$, $n \geq 1$. Let W be any averaging word with $\text{ar}_W = n + 1$. Let $w_1 \cdots w_m$ be the standard form of W , then we have w_i is either x or $\lfloor w'_i \rfloor$, $1 \leq i \leq m$. According to the breadth of W , we consider the following two subcases: (i) $m \geq 2$; (ii) $m = 1$.

- (i) If $m \geq 2$, then we have $\tau := \varphi(w_1 \cdots w_m) = t_\mu \circ (\varphi(w_1 \cdots w_{m-1}) \otimes \varphi(w_m))$. Denote the root of τ by v . Then $\tau_{v,l} = \varphi(w_1 \cdots w_{m-1})$, $\tau_{v,r} = \varphi(w_m)$. By the induction hypothesis, we have $\tau_{v,l}, \tau_{v,r} \in \text{Avt}$. Since each bracketed subtree of τ is a bracketed subtree of either $\tau_{v,l}$ or $\tau_{v,r}$, τ satisfies (a) for Avt .

Except for the root v of τ , each bi-vertex of τ is a bi-vertex of either $\tau_{v,l}$ or $\tau_{v,r}$. In order to show τ satisfies (b) for Avt , we only need to prove v satisfies (b) for Avt .

(α) If $w_m = x$, then $\tau_{v,r}$ is a trivial tree.

(β) If $w_m = \lfloor w'_m \rfloor$, then $\varphi(w_m)$ is a bracketed tree and $w_{m-1} = x$. Thus $\tau_{v,l}$ is trivial when $m = 2$ and $\tau_{v,l}$ has a trivial right subtree when $m \geq 3$.

We have τ satisfies (b) for Avt . Therefore, $\tau \in \text{Avt}$.

- (ii) If $m = 1$, then $W = \lfloor V \rfloor^s$, where $V \in \Omega^+$. Note that $\text{ar}_W \geq 2$ and by Remark 2.9. Rewrite W as the form of $\lfloor v_1 \cdots v_k \rfloor^s$. Then $k \geq 2$, $v_1 = x$, and $v_k = x$ or $v_k = \lfloor v'_k \rfloor$, $v'_k \in \Omega^+$. Let $\tau := t_p^s \circ \varphi(V) = \varphi(W)$.

Except for $\sigma_i := t_p^i \circ \varphi(V)$, $1 \leq i \leq s$, each bracketed subtree of τ is a bracketed subtree of $\varphi(V)$. We only need to prove σ_i is a fundamental averaging tree. Since $w_1 = x$ and $\text{pb}(\sigma_{i,r}) = \text{pb}(\varphi(v_k)) \leq 1$, σ_i is a fundamental averaging tree and τ satisfies (a) for Avt .

Since $V = v_1 \cdots v_k$ is an averaging word with $\text{ar}_V = n + 1$ and $k \geq 2$, (i) implies $\varphi(V) \in \text{Avt}$. It follows from each bi-vertex of τ is a bi-vertex of $\varphi(V)$ that τ satisfies (b) for Avt . Therefore, $\tau \in \text{Avt}$.

Step 2. Define a map $\Phi : \text{Avt} \longrightarrow \mathcal{A}(\{x\})$ such that $\varphi\Phi = \text{Id}_{\text{Avt}}$:

We do this by induction on lef_τ . When $\text{lef}_\tau = 1$, that is τ is either trivial or a ladder. Define

$$\Phi(\text{id}) = x \quad \text{and} \quad \Phi(t_p^\ell \circ \text{id}) = \lfloor x \rfloor^\ell.$$

We immediately have $\varphi\Phi(\text{id}) = \varphi(x) = \text{id}$ and $\varphi\Phi(t_p^\ell \circ \text{id}) = \varphi(\lfloor x \rfloor^\ell) = t_p^\ell \circ \text{id}$.

Assume that Φ has been defined for averaging trees τ with $\text{lef}_\tau \leq n$, $n \geq 1$ and $\varphi\Phi(\tau) = \tau$. For any averaging tree τ with $\text{lef}_\tau = n + 1$. Rewrite τ as $t_p^s \circ \tau'$ with $s \geq 0$ and τ' is non-bracketed. Then we have $\tau = t_p^s \circ (t_\mu \circ (\tau_l \otimes \tau_r))$ for some τ_l, τ_r , where τ_r is either trivial or bracketed. By Remark 4.1, τ_l and τ_r are still averaging trees. By the induction hypothesis, there exist averaging words W_1 and W_2 such that $\Phi(\tau_l) = W_1$, $\varphi(W_1) = \tau_l$ and $\Phi(\tau_r) = W_2$, $\varphi(W_2) = \tau_r$ respectively. By the definition of φ and the property of τ_r , we have the breadth of W_2 is 1. Define $\Phi(\tau) = \lfloor W_1 W_2 \rfloor^s$. Then we have

$$\begin{aligned} \varphi\Phi(\tau) &= \varphi(\lfloor W_1 W_2 \rfloor^s) = t_p^s \circ \varphi(W_1 W_2) \\ &= t_p^s \circ t_\mu \circ (\varphi(W_1) \otimes \varphi(W_2)) \quad (\text{the breadth of } W_2 \text{ is } 1) \\ &= t_p^s \circ t_\mu \circ (\tau_l \otimes \tau_r) = \tau. \end{aligned}$$

It remains to prove Φ is well-defined, that is we need to show $\lfloor W_1 W_2 \rfloor^s \in \mathcal{A}(\{x\})$.

In fact, if $s = 0$, by (b) for Avt, we have $W_2 = x$ or $W_2 = \lfloor V \rfloor$. If $W_2 = x$, then $W = W_1 x$ is still an averaging word. If $W_2 = \lfloor V \rfloor$, we have $\tau_r = t_p^\ell \circ \tau'_r$ is bracketed. First, W_1 is not the form $\lfloor V' \rfloor$, otherwise τ_l would be bracketed and then doesn't have right subtree. Second, if $W_1 = w_{11} w_{12} \cdots w_{1t}$, then w_{1t} is not the form $\lfloor V'' \rfloor$ (the tail of W_1), otherwise the right subtree of the root of τ_l would be bracketed (not trivial) and gives a contradiction. So we have $w_{1t} = x$ and $W = W_1 W_2$ is still an averaging word.

If $s > 0$, that is τ is bracketed. Let $W_1 = w_{11} w_{12} \cdots w_{1t}$ be the standard decomposition of W_1 . By (b) for Avt, we have either $W_2 = x$ or $W_2 = \lfloor V \rfloor^\ell$, where $V \in \Omega^+$. If $W_2 = \lfloor V \rfloor^\ell$ and $\ell \geq 2$, then τ_r is a bracketed tree with $\text{pb}(\tau_r) = \ell \geq 2$. Thus τ is not a fundamental averaging tree. It gives a contradiction with τ is an averaging tree. So we have $\ell = 1$.

Since τ is a bracketed tree, $\text{Lf}(\tau)$ is a fundamental averaging tree and bracketed, we have $w_{11} = x$, otherwise there would be a bi-vertex of τ whose left subtree is bracketed. Hence $W_1 W_2$ is an averaging word with the head x and W_2 is either x or a bracketed word of the form $\lfloor V \rfloor$, where $V \in \Omega^+$. That is $W_1 W_2 \in \Omega^+$. Therefore $\lfloor W_1 W_2 \rfloor^s$ is still an averaging word.

Step 3. Φ satisfies $\Phi\varphi = \text{Id}_{\mathcal{A}(\{x\})}$:

We prove it by induction on ar_W . When $\text{ar}_W = 1$, that is $W = x$ or $\lfloor x \rfloor^s$. Then we have $\Phi\varphi(x) = \Phi(\text{id}) = x$ and $\Phi\varphi(\lfloor x \rfloor^s) = \Phi(t_p^s \circ \text{id}) = \lfloor x \rfloor^s$.

Assume that $\Phi\varphi = \text{Id}_{\mathcal{A}(\{x\})}$ holds for any W with $\text{ar}_W \leq n$, $n \geq 1$. Let W be any averaging word with $\text{ar}_W = n + 1$. According to breadth of W , we consider the following two subcases:

(1) If $m \geq 2$ and $W = w_1 \cdots w_m$ is in standard form. We have

$$\begin{aligned} \Phi\varphi(w_1 \cdots w_m) &= \Phi(t_\mu \circ (\varphi(w_1 \cdots w_{m-1}) \otimes \varphi(w_m))) \\ &= \Phi(\varphi(w_1 \cdots w_{m-1}))\Phi(\varphi(w_m)) \quad (\text{by the definition of } \Phi) \\ &= w_1 \cdots w_{m-1} w_m. \quad (\text{by induction hypothesis}) \end{aligned}$$

(2) If $m = 1$, then $W = \lfloor V \rfloor^s$, where $V \in \Omega^+$. Rewrite W as the form of $\lfloor v_1 \cdots v_k \rfloor^s$. Then $k \geq 2$, $v_1 = x$, and $v_k = x$ or $v_k = \lfloor v'_k \rfloor$, $v'_k \in \Omega^+$. We have

$$\begin{aligned} \Phi\varphi(\lfloor v_1 \cdots v_k \rfloor^s) &= \Phi(t_p^s \circ \varphi(v_1 \cdots v_{k-1} v_k)) \\ &= \lfloor \Phi(\varphi(v_1 \cdots v_{k-1} v_k)) \rfloor^s \quad (\text{by the definition of } \Phi) \\ &= \lfloor v_1 \cdots v_k \rfloor^s. \quad (\text{by case (1)}) \end{aligned}$$

Therefore, φ is a bijection between $\mathcal{A}(\{x\})$ and Avt. □

Now, we can give the free averaging associative algebra in terms of trees.

Theorem 4.3. *The free averaging associative algebra on one generator is $\oplus_{n \geq 1} \mathbf{k}[\text{Avt}_n]$, where Avt_n is the set of averaging trees with n leaves. The binary operations on $\text{Avt}_p \times \text{Avt}_q$ and the averaging operator are given respectively by:*

$$\tau_p * \tau_q = \varphi(\varphi^{-1}(\tau_p) \diamond \varphi^{-1}(\tau_q)), \quad Q(\tau) = \varphi(P(\varphi^{-1}(\tau))).$$

It is cumbersome to describe the operations $*$ and Q directly in terms of averaging trees. So we will not provide it here. On the other hand, both Proposition 4.2 and Theorem 4.3 can be easily generalized to free averaging algebras on any nonempty set X . Just replace Avt by trees from Avt with leaves decorated by elements of X .

4.2. The operad of averaging associative algebras. The operad of the averaging algebra (also called the averaging operad) is given as a quotient of the free operad in [22]. We will recall its definition and apply the description of free averaging algebra on one generator in terms of unreduced binary trees in the previous subsection to give an explicit construction of the averaging operad.

Definition 4.4. Let $V = V_2$ be a graded vector space concentrated at arity 2.

- (a) [19, Section 5.8.5] Let V_P denote the graded spaces concentrated at arity 1 and arity 2 with $(V_P)_1 = \mathbf{k} \text{id} \oplus \mathbf{k} P$ and $(V_P)_2 = V$, where P is a symbol. Let $\mathcal{T}_{ns}(V_P)$ be the free operad generated by V_P .
- (b) [22] Let $\mathcal{P} = \mathcal{T}_{ns}(V)/(R_{\mathcal{P}})$ be a binary operad defined by generating operations V and relations $R_{\mathcal{P}}$. Let

$$AV_{\mathcal{P}} := \{ \omega \circ (P \otimes P) - P \circ \omega \circ (P \otimes \text{id}), \omega \circ (P \otimes P) - P \circ \omega \circ (\text{id} \otimes P) \mid \omega \in V \}.$$

Define the **operad of averaging \mathcal{P} -algebras** by

$$AV(\mathcal{P}) := \mathcal{T}_{ns}(V_P)/(R_{\mathcal{P}}, AV_{\mathcal{P}}),$$

where $(R_{\mathcal{P}}, AV_{\mathcal{P}})$ is the operadic ideal of $\mathcal{T}_{ns}(V_P)$ generated by $R_{\mathcal{P}} \cup AV_{\mathcal{P}}$.

- (c) Let As denote the non-symmetric operad of associative algebras. We call $AV(As)$ the **averaging operad**.

The operad $AV(As)$ is a non-symmetric operad and so is completely determined by the free nonunitary averaging associative algebra on one generator. So by Theorem 4.3, we have

Proposition 4.5. *Let $\mathcal{P}_0 = 0$ and $\mathcal{P}_n = \mathbf{k}\text{Avt}_n$. Then $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots)$ is the averaging operad.*

We end the paper by a description of the operad compositions of \mathcal{P} by making use of bijection φ in Proposition 4.2 together with the following fact: Let I_{AV} be the operated ideal of the free operated algebra $\mathbf{k}\mathfrak{S}(X)$ generated by elements of the form

$$[u][v] - [u[v]], \quad [[u]v] - [u[v]], \quad u, v \in \mathbf{k}\mathfrak{S}(X).$$

Then the quotient operated algebra $\mathbf{k}\mathfrak{S}(X)/I_{AV}$ is the free averaging algebra on X . Thus we have $\mathbf{k}\mathfrak{S}(X)/I_{AV} \cong \mathcal{A}(X)$ as operated algebras. Let

$$\text{Red} : \mathbf{k}\mathfrak{S}(X) \rightarrow \mathbf{k}\mathfrak{S}(X)/I_{AV} \cong \mathcal{A}(X)$$

be the composition of the quotient map with the above isomorphism.

Now let $\tau \in \mathcal{P}_m$, $\sigma \in \mathcal{P}_n$ and $1 \leq i \leq m$. To define the composition $\tau \circ_i \sigma$, first replace the i -th x of $\varphi^{-1}(\tau)$ (from the left) by $\varphi^{-1}(\sigma)$ and denote the result by W . Then we have $\tau \circ_i \sigma = \varphi(\text{Red}(W))$.

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